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Common Fixed Point And Best Approximation Results In Cone Metric Space

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Abstract: Fixed point theory has many applications in different branches of science. This theory itself is a beautiful mixture of analysis, topology, and geometry. In 2007, Huang and Zhang introduced the concept of cone metric space as a generalization of metric space, in which they replace the set of real numbers with a real Banachspace. The concept of almost contraction for multi-valued mappings in the setting of cone metric spaces is defined and then we establish some fixed point and common fixed point results in the set-up of cone metric spaces. As an application, some invariant approximation results are obtained. The results of this paper extend and improve the corresponding results of multi-valued mapping from metric space theory to cone metric spaces. The Authors proved several fixed point theorems for contractive type mappings on a cone metric space when the underlying cone is normal.

Keywords: Cone metric, fixed point, approximation, topology, geometry, Banach space.

INTRODUCTION

Huang and Zhang have introduced the concept of cone metric space, replacing the set of real numbers by an ordered Banach space. Fixed point theory has many applications in various branches of science. This theory itself is a beautiful blend of analysis, topology and geometry. Since the appearance of Banach contraction mapping theory, there has been a lot of activity in this field and many famous fixed point theorems came into existence as a generalization of that
theory. Many authors generalized and extended the notion of metric spaces for a useful discussion of these generalizations of metric locations such as B-metric spaces, partial metric spaces, generalized metric spaces, complex-valued metric spaces, etc., one can refer to the service.

Nadler and Mark in pioneered the study of fixed point theorems for multi-valued mapping and established the multi-valued version of the Bannach contraction mapping theory. Since the theory of multi-valued mapping has many applications, it became a center of research over the years. Recently, several authors worked out the results on multi-valued mapping defined at the cone metric space when the underlying cone is normal or regular.

The concept of approximate contraction is defined for multi-valued mapping in the setting of conic metric spaces and then we establish some fixed point and common fixed point results in the set-up of cone metric spaces. In this way our results extend the results of Arshad and Ahmed and also improve the concomitant results of both single-valued and multi-valued mappings present in the literature.

I. Let E be a real Banach space with norm and P be a subset of E. Then P is called a cone if -

- ✓ P is nonempty, closed, and $P \neq \{\theta\}$, where θ is the zero element of E;
- ✓ for any non-negative real numbers a, b and for any $x,y \in P$, one has $ax+by \in P$;
- ✓ $x \in P$ and $-x \in P$ implies $x=\theta$.

Given a cone P⊆EP⊆E, we define a partial ordering ⪯ with respect to P by x⪯yx⪯y if and only if $y-x \in P$. We shall write x≺yx≺y if x⪯yx⪯y and x≠yx≠y while x≪yx≪y if and only if $y-x \in intP$, where intP is the interior of P. A cone P is called normal if there is a number K>0K>0 such that for all $x,y \in E$
$\theta \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$.

The least positive number K satisfying the above inequality is called the normal constant of P. In the following we suppose that E is a real Banach space and P is a cone in E with $intP \neq \phi$ and ⪯ is a partial ordering with respect to P.

II. Let X be a nonempty set. Suppose the mapping $d: X \times X \to E$ satisfies the following:

(d_1): $\theta \preceq d(x,y)$ for all $x,y \in X$;
(d_2): $d(x,y)=\theta$ if and only if $x=y$;
(d_3): $d(x,y)=d(y,x)$ for all $x,y \in X$;
(d_4): $d(x,y) \preceq d(x,z)+d(z,y)$ for all $x,y,z \in X$.
Then d is called a cone metric on X, and (X,d) is called a cone metric space.

III. Let (X,d) be a cone metric space and let {xn} be a sequence in X. Then the sequence {xn} obeys the following.

- {xn} converges to x, if for every c∈Ec∈E with θ≪cθ≪c there exists a positive integer N such that $d(x_n,x) \ll c$, for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$.
- {xn} is said to be Cauchy if for every $c \in E$ with $\theta \ll c$ there exists a positive integer N such that $d(x_n,x_m) \ll c$, for all $n,m \geq N$.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

IV. Let P be a cone in Banach space E. Then the following properties hold:

a. If $c \in intP$ and $a_n \to \theta$, then there exists a positive integer N such that for all $n>N$, we have $a_n \ll c$.
b. If $a \preceq ka$, where $a \in P$ and $0 \leq k < 1$, then $a=\theta$.

V. Let (X,d) be a cone metric space and let C(X) be the family of all nonempty and closed subsets of X. A map $H:C(X)\times C(X)\to E$ is called a H-cone metric on C(X) induced by d if the following conditions hold:

- **(H_1):** $\theta\preceq H(A,B)$ for all $A,B\in C(X)$.
- **(H_2):** $H(A,B)=\theta$ if and only if A=B.
- **(H_3):** $H(A,B)=H(B,A)$ for all $A,B\in C(X)$.
- **(H_4):** $H(A,B)\preceq H(A,C)+H(C,B)$ for all $A,B,C\in C(X)$.
- **(H_5):** If $A,B\in C(X)$, $\theta\prec\epsilon\in E$ with $H(A,B)\prec\epsilon$, then for each $a\in A$ there exists $b\in B$ such that $d(a,b)\prec\epsilon$.

VI. Let (X,d) be a metric space. Then the mapping $H_u:C(X)\times C(X)\to R$ defined by $H_u(A,B)=\max\{\sup_{x\in A}d(x,B),\sup_{y\in B}d(y,A)\}$ is an H-cone metric induced by d. It is also known as the usual Hausdorff metric induced by d.

It is to be noted that (C(X),H) is a complete metric space whenever (X,d) is a complete metric space.

VII. Let X be a nonempty set, $T:X\to C(X)$ be a multi-valued mapping, and $f:X\to X$. Then an element $x\in X$ is said to be

(i) a fixed point of T, if $x\in Tx$;

(ii) a common fixed point of T and f, if $x=fx\in Tx$;

(iii) a coincidence point of T and f, if $w=fx\in Tx$, and w is called the point of coincidence of T and f.

We denote $C(f,T)=\{x\in X:fx\in Tx\}$, the set of coincidence point of f and T. The set of fixed point of T and the set of common fixed point of f and T is denoted by F(T) and F(f,T) respectively.

VIII. Let X be a nonempty set, $T:X\to C(X)$ be a multi-valued mapping, and $f:X\to X$. Then f is called T-weakly commuting at $x\in X$ if $ffx\in Tfx$.

IX. Let (X,d) be a cone metric space and let there exist an H-cone metric on C(X) induced by d. A map $T:X\to C(X)$ is said to be a multi-valued almost contraction if there exist two constants $\lambda\in(0,1)$ and $L\geq 0$ such that

$H(Tx,Ty)\leq\lambda d(x,y)+Ld(y,u)$ for all $x,y\in X$ and $u\in Tx$.

<u>Let us take an example on Theorem 10</u>

Let $X=[0,1]$, $E=C^1_R([0,1])$ with the norm $\|\varphi\|=\sup_{x\in X}|\varphi(x)|+\sup_{x\in X}|\varphi'(x)|$ and consider the cone $P=\{\varphi\in E:\varphi(t)\geq 0\}$. Suppose $\varphi,\phi\in E$ are defined as

$\varphi(x)=x$ and $\phi(x)=x^{2n}$ for each $n\geq 1$

Then θ⪯ϕ⪯φand ‖φ‖=2, ‖ϕ‖=2n+1. Given any K>0 we can find a positive integer n such that 2n+1>2K. So, ‖ϕ‖≰K‖φ‖ for any K>0. Thus, P is non-normal cone. Now, define d:X×X→E by
d(x,y)=|x−y|φ,

where φ:[0,1]→R with φ(t)=et. Then (X,d) be a complete cone metric space. Let C(X) be the family of all nonempty and closed subsets of X and define a mapping H:C(X)×C(X)→E as
H(A,B)=Hu(A,B)φfor all A,B∈C(X),

where Hu is the usual Hausdorff metric induced by d(x,y)=|x−y|. Also define T:X→C(X)by
T(x)={[0,x2][23,x3+12]for x∈[0,12],for x∈[12,1]}.

Now we shall show that T is a multi-valued almost contraction, that is, we show that T will satisfy condition (2.1). For this, we consider the following possible cases:

Case (1). If x∈[0,12] and y∈(12,1], then condition (2.1) can be written as
|||x2−(y3+12)|||et≤λ|x−y|et+L|y−u|et|(2.9)Forall u∈Tx=[0,x2]. Here, we observe that |x2−(y3+12)|≤56|, |x−y|∈(0,1], and |y−u|>14 for all u∈[0,x2]. Thus, the inequality (2.9) is true for any λ∈(0,1) and L≥103.
Case (2). If x∈(12,1] and y∈[0,12], then condition (2.1) takes the form
|||(x3+12)−y2|||et≤λ|x−y|et+L|y−u|et|(2.10)for all u∈Tx=[23,x3+12].
Inthis case |(x3+12)−y2|≤56, |x−y|∈(0,1) and |y−u|≥16for all u∈[23,x3+12]. Thus, the inequality (2.10) is true for any λ∈(0,1) and L≥5.
Case (3). If x,y∈[0,12], then
H(Tx,Ty)===≤Hu(Tx,Ty)φHu([0,x2],[0,y2])et12|x−y|etλd(x,y)+Ld(y,u)

for any λ∈[12,1) and L≥0, where u is arbitrary element of Tx.
Case (4). If x,y∈(12,1), then
H(Tx,Ty)===≤Hu(Tx,Ty)φHu([23,x3+12],[23,y3+12])et13|x−y|etλd(x,y)+Ld(y,u)

for any λ∈[13,1) and L≥0, where u is arbitrary element of Tx.
Now, from all the cases, it is concluded that the multi-valued mapping T satisfies the inequality (2.1) for λ=12 and L=5. Hence, T is an almost multi-valued contraction that satisfies all the hypotheses of Theorem 10. Thus, the mapping T has a fixed point. Here x=0is such a fixed point.

Since the presence of Minardus' result in the best estimation theory, many authors have obtained the best approximate results for single-valued maps as an application of fixed point and general fixed point results.The best estimation results for multi-valued mapping were obtained by Kamran, Al-Thaga and Shehzad, Baig et al., O'Regan and Shehzad, and Markin and Shehzad. In addition, the best approximation result in establishing the cone metric space was considered for the result time.

The following theorem ensures the existence of a fixed point from the set of best approximations.

Theorem 23 Let M be a subset of a cone metric space (X, d) and let there exist an H-cone metric on C(X) induced by d. Suppose T : X ^ C(X), p e X, and BM(p) is nonempty, compact, and it has a joint contractive family F = {hA : A e C(BM(p))}. If T is continuous on BM(p), (3.3) holds for all x, y e BM(p) and also d(y,p) < d(x,p) for all x e BM(p) andy e Tx. Then BM(p) n F(T) = 0.

Proof- We claim Tx c BM(p) for each x e BM(p). To prove this let x e BM(p), then d(x,p) ^ d(z, p) for all zeM .If u e Tx, then by the given hypothesis

d(u, p) ^ d(x, p) ^ d(z, p) for all zeM.

Thus, u e BM(p). So T : BM(p) ^ C(BM(p)) is a multi-valued mapping and hence by applying Theorem 22 for Bm(p), it follows that Bm(p) n F(T) = 0.

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