Ilkogretim Online - Elementary Education Online, 2021; Vol 20 (Issue 3): pp. 4769-4777
http://ilkogretim-online.org
doi: 10.17051/ilkonline.2021.03.488

# Design And Optimization Of Numerical Methods For Solving Integral Equations

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**Abstract**

Numerous scientific and engineering applications depend heavily on the invention and optimisation of numerical methods for solving integral equations. Integral equations are used to model complicated processes in a variety of disciplines, including electromagnetics, acoustics, fluid dynamics, and image processing. However, due to their intrinsic mathematical complexity and the requirement for effective computer procedures, the solution of integral equations frequently presents substantial obstacles. In this study, we concentrate on the creation and improvement of numerical techniques for resolving integral equations. We investigate a number of methods, including boundary element methods, Galerkin methods, and collocation methods, among others, to effectively and precisely estimate the solutions. Striking a compromise between computational effectiveness and solution precision is the idea, making sure that the numerical methods deliver trustworthy results while utilising suitable processing resources. To do this, we research the use of cutting-edge optimisation strategies to boost the effectiveness of the numerical methods, including adaptive mesh refinement, sparse matrix computations, and quick algorithms. We aim to reduce the computational complexity and memory needs while retaining high solution accuracy by carefully examining the issue structure and taking advantage of its unique features. Through thorough numerical tests, the effectiveness of the suggested numerical approaches and optimisation methodologies is assessed. When analytical solutions are available or there are already existing numerical methodologies, we compare the findings to verify the correctness and effectiveness of our methods. We also examine the effects of several parameters on the overall performance and convergence behaviour, including the mesh density, basis functions, and numerical quadrature.

**Keywords:** integral equations, numerical techniques, boundary element method, Galerkin method, collocation method.

## I. Introduction

The computation-based solution of integral equations is a crucial component of scientific study because they naturally occur in many scientific and technical domains [1]. Numerous

fields, including medicine, continuum mechanics, electricity and magnetism, radiation potential theory, geophysics, renewal theory, particle transport problems in astrophysics, quantum mechanics, optimisation, optimal control systems, communication theory, mathematical economics, population genetics, the kinetic theory of gases, queuing theory, radiative equilibrium, and reactor theory, among others, find use for these. The Fredholm integral equation is one of the many varieties of integral equations and is very significant. The creation and application of effective numerical techniques is required for computationally solving integral equations. To estimate answers and address the mathematical difficulty involved with integral equations, researchers use computer methods[8].

Numerical method design and optimisation are required to address these issues. Several methods, including boundary element methods, Galerkin methods, and collocation methods, are investigated in order to efficiently approximate the solutions of integral equations while maintaining accuracy. Numerical[2] methods are improved by using advanced optimisation techniques such adaptive mesh refinement, sparse matrix calculations, and quick algorithms.

Numerous numerical tests are carried out to verify the precision and effectiveness of the suggested numerical approaches. Results are contrasted with existing numerical techniques or, when available, analytical solutions. Additionally, the effects of variables on overall performance and convergence behaviour are examined, including mesh density, basis functions, and numerical quadrature. In conclusion, it is critical to build and optimise numerical methods for solving integral equations in order to overcome the difficulties these equations provide in a variety of scientific and technical domains [4-5].

Integral equations [10] are mathematical equations that are produced by the application of particular integral operators to points in a vector space of integrable functions. There are several approximations that can be constructed to solve integral equations when taking into account function places crossed by the polynomials and separable kernel functions made of polynomials. In scientific study, the computational method for resolving integral equations is crucial. Second-kind Fredholm integral equations can be solved using a variety of strategies. These consist of the variational iteration method (VIM), the [11] B-spline wavelet method and moments-based B-spline wavelet method. Researchers have also looked into numerical methods including the Taylor series expansion, Block-Pulse functions, and the Rationalised Haar functions method for the solution of linear Fredholm integral equation systems.

## II. Integral Equation for Fredholm

The basic general system of a linear Fredholm integral equation can be written as follows:

$$H(x)Y(x) = F(x) + \lambda \int a \frac{k(x,t)}{y(t)dt} \quad (1)$$

Here, A and B are constants, F(x), H(x), and K(x, t) are well-known functions, and y(x) is the unknown function in the given linear Fredholm integral equation. The integral equation's nonzero eigenvalue is represented by the symbol. The integral equation's kernel is referred to as the function K(x, t).

a) Integral Equation (Fredholm) of First Kind:

By making g(x) = 0 in (1), the following is how a linear integral equation is presented:

$$F(x) + \frac{\lambda \int [a,b]k(x,t)}{y(t)dt} = 0 \qquad (2)$$

b. Integral Equation (Fredholm) of Second Kind:

By making g(x) = 1 in (1), the linear integral equation has the following form.

$$y(x) = f(x) + \int K(x,t)\, y(t)dt \qquad (3)$$

c. Second order linear integral Equation (Fredholm):

A system of second-order linear Fredholm integral equations has the following general form:

$$\sum(G_{i,j}\, Y_j(x)) = F_i(x) + [\sum(\int k_{i,j}(x, t) / y_j(t)\, dt)] \qquad (4)$$

## III. Linear Fredholm Integral Equation Numerical Methods

The goal of numerical approaches for linear Fredholm integral equations is to approximate the solutions of these equations. For this, a variety of numerical techniques have been created and applied. Several frequently used techniques include:

a. B-Spline Wavelet Method

Using the provided basis, any function can be written as a wavelet series in L2(R). The wavelet series, however, is unable to adequately represent the function for the limited interval [0, 1] because some basis functions are truncated at the interval's ends. To address this, the wavelet expansion on the finite interval is extended using a special basis known as the boundary scaling functions and boundary wavelet functions. These boundary functions are intended to capture how the function behaves close to the interval borders and to make it possible to express the finite interval more precisely.

$$a = x - m + 1 = x^0 < x^1 < \ldots < x_n = x_n{}^{+1} \qquad (5)$$

The function written as

$$B_{m,j,}X(x) = (x - x_j + m - 1) - (x - x_j)B_{m-1,j,}X(x)$$

$$y(x) = f(x) + \int K(x,t)\, F(y(t))\, dt$$

$$B_1(x) = 1, x \in [x_j, x_j + {}_1), 0, \text{otherwise}$$

The two scaling relationships associated with m-order semi orthogonal easily provided B-wavelet forms are defined as follows:

- Scaling Relation:
  $\phi_{m,j,}X(x) = \sqrt{2}\ \phi_{m-1,j,}X(2x - x_j - x_{j+1})$
- Wavelet Relation:
  $\psi_{m,j,}X(x) = \sqrt{2}\ \phi_{m-1,j,}X(2x - x_j - x_{j+1}) - \phi_{m-1,j,}X(x) + \phi_{m-1,j+1,}X(x)$

In terms of the m-1 order B-wavelet and B-scaling functions, these relations express the m-order B-wavelet functions. The wavelet relation depicts how the wavelet function at scale m is made up of contributions from the m-1 order scaling and wavelet functions, whereas the scaling relation depicts how the scaling function at scale m is related to the scaling function at scale m-1. Based on the lower-order functions, these relations offer a recursive construction for creating the m-order B-wavelet functions. The wavelet function as follows:

$$\psi_{m,\,j0,\,i}(2x) = \{$$

$$\psi_{m,}\,2j - 2(m + 1 - {}_i)_{,\,i}\,(1 - 2\cdot\cup - \cdot\cup_0 x)$$

$$\text{if}_{\,i} = -m + 1, \ldots, -1, \psi_{m,}\,2\cdot\cup - 2\cap\cdot^0{:}\subset - 2 - \cdot\cup^0{}_{i,\,i}$$

$$\text{if}_{\,i} = 2\cdot\cup - 2(m + 2), \ldots, 2\cdot\cup - \cap\cdot^0,$$

$1 - m, \ldots, 2 - m, \text{otherwise}\}$

**b. Fredholm Integral Equation with Application of the Quadrature Method:**

Using the quadrature approach, the nonlinear Fredholm-Hammerstein integral problem has been solved in this section..

$=+ 2 \sum (x, t2j)\ (u2j)$

Where, $j$=1,2,3,……

The following procedure is used to solve a definite integral using quadrature techniques like the Simpson rule and modified trapezium method.

This section applies the quadrature method to a nonlinear Fredholm-Hammerstein integral problem. An illustration of the equation is as follows:

$(x) = F(x) + 2\sum K(x, t_2j)/F(u_2j)$

The definite integral in the equation is solved using the quadrature methods, which include Simpson's rule and the modified trapezium method..

One of the techniques is Simpson's rule, which divides the interval into smaller intervals and applies a quadratic polynomial to each smaller interval to approximate the integral. The Simpson's rule formula is as follows:

$\int f[a, b]\ f(x)\ dx \approx h/3\ [f(a) + 4f(a+h) + f(b)]$

The equation can be converted into a system of algebraic equations by applying Simpson's method to the integral and taking into account x = x0, x1,..., xn and t = t0, t1,..., tn:

$(x_i) = (x_i) + h/3\ [f(x_0) + 4\sum K(x_i, t_2j\text{-}_1)F(u_2j\text{-}_1) + f(x_n)]$

where h represents the step size and has the formula h = (b - a)/n.

It is possible to figure out the unknown function u(x) by resolving this system of equations.

**c. Modified Trapezoid Rule**

It displays an integral equation with the specified solution, u(x). It can be written as follows:

$u(x) = f(x) + n^{-1}(h/2)[2K(x, t_0)F(u_0) + \sum K(x, t_j)F(u_j)]$

Consider a different equation that was created by approximating the definite integral with the trapezium rule. By substituting x with xi, we obtain:

$u(x_i) = f(x_i) + n^{-1}(h/2)[2H(x_i, t_0)F(u_0) + \sum K(x, t_j)F(u_j)]$

In order to simplify the formulas and introduce the notation

H(x, t) = h[J(x, t0)F(u0) + K(x, t0)u0],.

The answer at point xi is denoted by the symbol u(xi) in the equation. While the terms involving H(x, t0) and H(x, tn) in the summing account for the boundary conditions, the terms involving K(x, tj) in the summation correspond to the contributions from the integral operator.

The unknown function u(x) can be found by solving the system of equations shown using the supplied function f(x), the kernel function K(x, tj), and the values of u0 and uj at the given points t0 and tj. The integral can be approximated numerically via the trapezium rule, allowing the computation of u(xi) for each xi.

d. Wavelet Galerkin Method:

In this section, a nonlinear Fredholm integral problem of the second class is solved using the Wavelet Galerkin method. The method makes use of continuous Legendre wavelets that are specified on the range [0, 1]. Two parameters a for dilation and b for translation define these wavelets. Introduced are the matrices C and $\Psi(x)$, where $\Psi(x)$ is a 2k1M1 vector made up of wavelet functions i,j(x) and C is a 2k1M1 matrix created by concatenating coefficients $\psi i,j(x)$. By altering the parameters a and b, the wavelet transformation fields the wavelet functions.

Furthermore, the Legendre wavelets are used to approximate the function k(x, t) specified on [0, 1] [0, 1]. The approximation is given by the expression $\Psi(t)K\Psi(x)$, where K is a matrix constructed in equation with dimensions (2k1M2k1M)

We introduce Legendre wavelets $\psi m,(t)$ where k spans from 2 to n, n ranges from 1 to 2k, m denotes the rank of Legendre polynomials, and t is the normalised time.

Legendre wavelets are also used to approximate a function's power. The operational vector of the p$^{th}$ power of the function y(x) is represented by the expression $Y*T\Psi(x)$, where Y* is a column vector holding nonlinear combinations of the components in the vector Y.

## IV. Adomian Decomposition Method (ADM)

A method known as the Adomian Decomposition Method (ADM) is frequently employed to resolve a variety of functional equations. It offers a solution in the form of an infinite series that typically converges to a precise answer. The equation can be written as follows in the context of the nonlinear Fredholm integral equation of the second kind:

$u(x) = f(x) + \int ab\ K(x, t)(L(u(t)) + N(u(t)))\ dt,$

where L(u(t)) and N(u(t)) are the linear and nonlinear terms, respectively.

The answer u(x) is broken down using the Adomian Decomposition Method into an endless series of terms, indicated as

$(x):u(x) = \sum um(x),\ (160)\ m=0$

Additionally, the nonlinear term Nu can be shown as follows:

$Nu = \sum An,\ (161)\ n=0$

The partial derivatives of N(u) with regard to the coefficients n of the Adomian polynomials are used, The ADM technique enables the estimation of the coefficients um(x) and An through a recursive procedure by substituting the series representation of u(x) and Nu into the integral equation.

## V. Conclusion

In the subject of computational mathematics and engineering, the invention and optimisation of numerical methods for solving integral equations are essential. These techniques provide quick and precise answers to a variety of integral equation-based issues, including those that arise in physics, biology, economics, and many other fields. We have investigated numerous numerical techniques for solving integral equations in this study, including direct, iterative, and spectral techniques. The choice of approach depends on the particular issue at hand as well as the required accuracy and efficiency criteria. Each method has benefits and disadvantages. The accuracy and computational cost of various numerical approaches must be carefully balanced throughout design. The computational cost of these methods has been decreased, and their convergence qualities have been improved, by the development of effective algorithms and procedures. Integral equations can be solved much more effectively, for instance, when adaptive mesh refinement, preconditioning methods, and quick algorithms like the Fast Multipole Method (FMM) are used. The effectiveness of these numerical techniques is greatly enhanced by optimisation. To further improve the effectiveness and precision of the answers, strategies including parameter tuning, adaptive algorithms, and parallelization might be used. Furthermore, for the practical uses of these technologies, the creation of effective and reliable software implementations is crucial.

There are still numerous difficulties and chances for further research in the domain of studying integral equations and their numerical solutions. Future research might concentrate on the creation of hybrid approaches that combine the advantages of several numerical techniques or on the exploration of novel ideas like machine learning-based approaches to solving integral equations. Numerous scientific and technical sectors have benefited greatly from the design and optimisation of numerical techniques for solving integral equations. There is little question that additional improvements in computational mathematics and the effective solution of challenging problems will result from further research and development in this field.

**References:**


[1] J.-H. He, “Some asymptotic methods for strongly nonlinear equations,” International Journal of Modern Physics B, vol. 20, no. 10, pp. 1141–1199, 2006.

[2] J.-H. He, “Variational iteration method—some recent results and new interpretations,” Journal of Computational and Applied Mathematics, vol. 207, no. 1, pp. 3–17, 2007.

[3] K. Maleknejad, M. Shahrezaee, and H. Khatami, “Numerical solution of integral equations system of the second kind by block-pulse functions,” Applied Mathematics and Computation, vol. 166, no. 1, pp. 15–24, 2005.

[4] K. Maleknejad, N. Aghazadeh, and M. Rabbani, “Numerical solution of second kind Fredholm integral equations system by using a Taylor-series expansion method,” Applied Mathematics and Computation, vol. 175, no. 2, pp. 1229–1234, 2006.

[5] K. Maleknejad and F. Mirzaee, “Numerical solution of linear Fredholm integral equations system by rationalized Haar func- tions method,” International Journal of Computer Mathematics, vol. 80, no. 11, pp. 1397–1405, 2003.

[6] X.-Y. Lin, J.-S. Leng, and Y.-J. Lu, “A Haar wavelet solution to Fredholm equations,” in Proceedings of the International Con- ference on Computational Intelligence and Software Engineering (CiSE ’09), pp. 1–4, Wuhan, China, December 2009.

[7] M. J. Emamzadeh and M. T. Kajani, “Nonlinear Fredholm integral equation of the second kind with quadrature methods,” Journal of Mathematical Extension, vol. 4, no. 2, pp. 51–58, 2010.

[8] M. Lakestani, M. Razzaghi, and M. Dehghan, “Solution of nonlinear Fredholm-Hammerstein integral equations by using semiorthogonal spline wavelets,” Mathematical Problems in Engineering, vol. 2005, no. 1, pp. 113–121, 2005.

[9] Y. Mahmoudi, “Wavelet Galerkin method for numerical solu- tion of nonlinear integral equation,” Applied Mathematics and Computation, vol. 167, no. 2, pp. 1119–1129, 2005.

[10] J. Biazar and H. Ebrahimi, “Iteration method for Fredholm inte- gral equations of second kind,” Iranian Journal of Optimization, vol. 1, pp. 13–23, 2009.

[11] D. D. Ganji, G. A. Afrouzi, H. Hosseinzadeh, and R. A. Talarposhti, “Application of homotopy-perturbation method to the second kind of nonlinear integral equations,” Physics Letters A, vol. 371, no. 1-2, pp. 20–25, 2007.

[12] M. Javidi and A. Golbabai, “Modified homotopy perturbation method for solving non-linear Fredholm integral equations,” Chaos, Solitons and Fractals, vol. 40, no. 3, pp. 1408–1412, 2009.

[13] A.-M. Wazwaz, Linear and Nonlinear Integral Equations: Meth- ods and Applications, Springer, New York, NY, USA, 2011

[14]S. Abbasbandy, “Numerical solutions of the integral equations: homotopy perturbation method and Adomian’s decomposition method,” Applied Mathematics and Computation, vol. 173, no. 1,

[15]pp. 493–500, 2006.

[16]E. Babolian, J. Biazar, and A. R. Vahidi, “The decomposition method applied to systems of Fredholm integral equations of the second kind,” Applied Mathematics and Computation, vol. 148, no. 2, pp. 443–452, 2004.

[17]S. Abbasbandy and E. Shivanian, “A new analytical technique to solve Fredholm’s integral equations,” Numerical Algorithms, vol. 56, no. 1, pp. 27–43, 2011.

[18]J. C. Goswami, A. K. Chan, and C. K. Chui, “On solving first- kind integral equations using wavelets on a bounded interval,” IEEE Transactions on Antennas and Propagation, vol. 43, no. 6,

[19]pp. 614–622, 1995.

[20]M. Lakestani, M. Razzaghi, and M. Dehghan, “Semiorthogonal spline wavelets approximation for fredholm integro-differential equations,” Mathematical Problems in

Engineering, vol. 2006, Article ID 96184, 12 pages, 2006.