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# State of the art on the teaching of linear algebra in engineering careers.

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**ABSTRACT-** This article presents a selection of works considered relevant for the teaching of linear algebra through plausible reasoning both in engineering careers and in other research areas. We considered works in which an exhaustive study is made on the teaching of algebra through the North American school, others from the European school and another selection of schools from the rest of the world, then a small selection of works that are related to plausible reasoning in the teaching of mathematics is made, only one work is found where the teaching of linear algebra is related to plausible reasoning.

**KEYWORDS: State of the art, linear algebra teaching, plausible reasoning.**

## I. INTRODUCTION

Linear algebra constitutes an essential element in the basic training of an engineer, since in addition to being a basic science subject, it has immense applications in physics, statistics, differential equations, function modeling, circuits, computer algorithms, cryptography, etc. Therefore, it is important that engineering students develop concepts rather than techniques; of matrices, vectors, bases, subspaces, linear applications, vectors and eigenvalues, aspects that are closely linked to the solution of problems related to their profession; in addition, these concepts provide the possibility of a clear and precise advancement of their knowledge applicable in subsequent courses and courses of their specific knowledge.

Studies carried out since the 90s of the last century show that the learning of linear algebra presents great difficulties in students and therefore research has been carried out to detect the causes and propose solutions to the problem; from these we can summarize the following aspects.

a) Lack of active student participation, since the teacher's teaching is focused on the acquisition of manipulative and computational skills, which reduces student motivation.

b) Memoristic learning is used, based on the repetition of a series of procedures, a set of algorithmic recipes of totally mechanical application.

c) There is the obstacle of formalism, as Jean-Luc Dorier (2002) calls it, in which teaching is based on demonstrative reasoning and there is no room for plausible reasoning, conjecture or intuition, since the former has rigid models, codified and clarified by logic (formal or demonstrative), which is the theory of demonstrative reasoning.[1].

d) There are authors, besides Jean-Luc Dorier (2002), such as Sierpinska (2000), Carlson (1993), Harel (1989), who have been researching the subject for decades and who affirm that, since this is a problem of mathematics education, there is no formula that solves the problem, but that each research that is carried out is a contribution that enriches and therefore improves its teaching.

e) Many of the investigations that have been carried out for the improvement of teaching and learning of linear algebra have to do with geometric interpretations, including the use of algebraic and geometric computer systems such as Derive, Cabri, Geogebra, Matlab, among others. Some works have designed methodological strategies based on constructivism, such as the so-called APOE theory (Action-Process-Object-Scheme), which consists of describing the constructions and mental mechanisms

[1]Polya G. (1966). *Matemáticas y razonamiento plausible*. Editorial Tecnos, S. A. Madrid.

(internalization, coordination, encapsulation, assimilation) that a student can carry out to elaborate a given mathematical concept.

From the review of the research conducted to date, it has not been possible to identify any research based on a quasi-empirical approach.

Furthermore, in mathematics courses taught at the higher level, there are not many spaces that allow the development of plausible reasoning to develop the classes. A reflection on this situation and its importance in mathematics education has been referred to by Gascón (2000). The aforementioned characterizes a traditional mathematics class, in particular that of Linear Algebra (LA), as teaching-learning spaces where the student does not experience active methods in the construction of his learning from the argumentative point of view of the contents and therefore there is a lack of being able to appreciate other forms of validation, since the only one presented is the formal development carried out by the teacher.

One of the main problems in the teaching of vector spaces lies in "formalism", a conclusion reached by many of the researches on the teaching of linear algebra. Some of these works have tried to approach the problem from the intuitionistic didactic point of view, others from the geometric point of view. Some say that the problem is semiotic, since students are presented with the definitions formally, without having clarified some semantic aspects of them.Others say that the issue is that students have minimal formal logic bases and this makes the concepts more complicated for them. But one of the main researchers, who the author of this thesis considers stands out for his results on the subject is Jean-Luc Dorier, who argues that there is no single valid formula to solve these problems, since the cognitive processes of mathematics are too complex. Therefore, all the research carried out to improve the difficulties in the processes of teaching and learning mathematics should help teachers to make their teaching richer, more expert and more flexible, which is precisely what this work intends to do.

In this work we hope to investigate the studies on the teaching and learning of linear algebra in engineering careers and, based on this, to present a methodological contribution that describes a didactic model based on teaching materials supported by plausible reasoning, the use of technology, geometric visualization and problem solving in engineering careers.

## II. METHODOLOGY

An analysis was made of the main research that has been carried out on the problems and proposals for teaching and learning linear algebra in engineering and other areas, with the use of technology, as well as on the teaching of mathematics through plausible reasoning and mathematical quasi-empiricism.

a) **Methodology applied.** This study is developed under a qualitative approach, where a documentary review of the state of the art in some databases is carried out.

b) **Procedure.** In order to carry out this work, a search was made for research on methodological proposals, recommendations, problems in the teaching and learning of linear algebra in engineering careers and other areas, as well as research on the teaching of mathematics based on the works of Polya and Lakatos.

For the analysis of scientific articles, the study focused on their activity and impact. In the activity indicators, the current state of science was visualized and within this, the number of publications and their productivity were analyzed. For the impact indicators, the number of citations of the research was taken into account, which gives a characterization of the importance of the document and the recognition given to it by the researchers.

## III. RESULTS AND DISCUSSION

### On the teaching of linear algebra

**1.** Linear Algebra Currículum Study Group (LACSG).

During the last decade of the last century in the United States, a reform movement in the teaching of linear algebra at the university level was formed. Some mathematics professors concerned about improving the teaching and learning processes of linear algebra formed an organization called Linear Algebra Curriculum Study Group (LACSG).

Thus the group formed a few years later, together with the Mathematical Association of America (MAA), published the first study, where they collected a series of papers on aspects of teaching and learning of linear algebra grouped into five categories: (1) the role of linear algebra, (2) linear algebra from the point of view of the other disciplines where it is taught, (3) teaching linear algebra, (4) methodology of teaching linear algebra, and (5) applications of linear algebra.

One of the founders of this group is David Carlson who approaches the problem of teaching and learning linear algebra from the perspective of a mathematician by asking: How do we mathematicians learn, and answers that it all starts with an initial stimulus, an article or a conversation, then we work with examples, make conjectures, solve problems, do demonstrations and, finally, we communicate with our colleagues, both in writing and orally.[2].

A constructivist but also quasi-empiricist approach. He suggests a Piaget-focused learning where students learn how to learn, that is, that the teacher teaches students to learn mathematics on their own. One of the tasks a mathematics teacher must perform is to encourage students to take responsibility for their own learning, but this is achieved only if it is done with hard work and effectively. Thus, following up properly on the independent work (as it is currently called when talking about credits here in Colombia) that students do after each class, with the material covered in it, will be very important, perhaps even more important than the class itself.

But this is not an easy task, since stimulating students in this direction is very difficult, and for this purpose problem solving is proposed as a strategy; as a consequence of this, the group proposes the problem solving approach as a fundamental role in learning. Another researcher in this group is Carl Cowen, professor of mathematics at Indiana University, who states that "in addition to problem solving, another teaching strategy that has been experimented in linear algebra courses is the use of projects. The proposals in this direction range from the proposal of a specific project to the use of a collection of projects to be developed throughout the semester". Another member of this group is Guershon Harel who proposes several principles for teaching and learning linear algebra, as will be seen in his article below. The basis of the recommendations of this group are based on:

- The group members' knowledge is informed by research on how students learn and how mathematics should be taught, as well as epistemological and pedagogical considerations involving the learning and teaching of linear algebra. For example, the group recommends a strong emphasis on geometry since its interpretations will contribute significantly to student understanding. La experiencia que cada uno de los integrantes del grupo ha tenido en la enseñanza y aprendizaje del álgebra lineal.
- Consultation with the disciplines served by the course, their views on the curriculum and how it could be improved.

G. Harel, in his article Three principles of learning and teaching. Particular Reference to Linear Algebra - Old and New Observations proposes the need for an intellectually challenging course, with emphasis on demonstrations to improve understanding. For mathematics students a second course is proposed, while it is recommended to use technology in the first course of linear algebra. The author suggests a theoretical framework with three principles at the core of teaching and learning: the principle of concreteness, the principle of necessity and the principle of generalization.

In the principle of concreteness, the author suggests that linear algebra texts do not accommodate the pedagogical needs of students, since he considers that they are too abstract for the thinking that students at this level possess. For example, students can determine when a set of vectors is linearly independent in Rn, but they find it difficult when the question is asked of a set of functions. The premise of the concreteness principle is that students must construct understanding of a concept in a context that is concrete to them.

The second principle of necessity states: "for students to learn they must see the necessity of what is being taught. By necessity is meant an intellectual need rather than a social or economic need."

The third principle is that of generalization which states: when instruction refers to a "concrete" model, i.e., a model that satisfies the principle of concreteness, activities within this model should allow and induce generalization of concepts. This principle is intended to allow students to abstract concepts they learn in a specific model.

[2] Carlson, D. (1993). The linear algebra curriculum study recommendations for the first course in linear algebra. Recuperado del URL:http://www.jstor.org/discover/10.2307/2686430?uid=2&uid=4&sid=21104423314191

Another important professor is D. Carlson, who in his article: The linear algebra curriculum study recommendations for the first course in linear algebra refers to the curricula that linear algebra programs taught in non-mathematics majors, such as engineering, should have, saying that many do not adequately address the needs of students in these disciplines. He also calls attention to the difficulty students encounter in appropriating linear algebra concepts such as subspaces, generating set, and linear independence. When students have to learn these concepts they become confused and disoriented. The author says: "As if a dense fog covers them and they cannot see where they are or where they are going". The question then arises: ¿Why are these concepts so difficult for students?Carlson identifies the following reasons:

1) Linear algebra is taught very early on,
2) These are concepts, not computational algorithms,
3) Different algorithms are required to work with these ideas in different contexts, and
4) These concepts are introduced without substantial connection to students' prior experiences and without meaningful examples and applications.

He recommends that understanding the central ideas of linear algebra requires hard work, persistence and attention on the part of the student, no matter what the teacher's job is.

2. The European School

The European school is led by Jean-Luc Dorier who has a research group in mathematics education located in his home country France. Dorier presented his doctoral thesis from the approach of French mathematics didactics, dominated by Brousseau and Chevallard, on the teaching of the first concepts of linear algebra at the university level.

This group focuses its work on history and epistemology; its point of support is to work on the genesis of linear algebra and its learning, they are the creators of the notion of the "obstacle of formalism" and their research is based on how to avoid this obstacle. Dorier's group takes some considerations from the LACSG group, such as the three Harel's principles.

One of the principles of this group is to work in constructivism from the approach of French didactics and in particular didactic engineering, since they believe that this is another fundamental tool to solve the problem of the so-called obstacle of formalism. Likewise, they speak of the "goal lever" as a fundamental principle for learning mathematics, in which a support tool must be used at the right moment to, for example, solve a problem and make a mathematical reflection of what has been learned.

It is reported that numerous studies have been carried out to detect problems on the teaching and learning of linear algebra and likewise numerous suggestions have been made to try to solve them, but it is rightly argued that they are local and partial solutions. The most important thing is to continue working on it, since each investigation enriches the teaching practice of those who carry it out.

Jean L. Dorier in his article "Teaching Linear Algebra at University "addresses the main issues on the teaching of linear algebra which for the author are:

- The epistemological specificity of linear algebra and the interaction with research in the history of mathematics.
- Cognitive flexibility in the role of linear algebra learning.
- The three principles for the teaching of linear algebra as postulated by G. Harel.
- The relationship between geometry and linear algebra.
- An original teaching design experimented by M. Rogalski.

For him, two main traditions in the teaching of linear algebra can be distinguished: one focuses on the study of formal vector spaces while the other approach is more analytical with a focus based on the study of Rn and matrix calculus; however, the teaching of linear algebra is universally recognized as difficult. Students usually feel that they land on another planet, that they are overwhelmed by the number of new definitions and the lack of connection with previous knowledge. On the other hand, teachers often feel frustrated and disarmed when confronted with their students' inability to cope with ideas they consider to be so simple. Usually, it is blamed on the lack of mastery in basic logic and set theory or on the students' inability to use geometric intuition.

For Dorier, an epistemological analysis of the history of linear algebra is a way to reveal some possible sources of student difficulties. Several works have been carried out in this direction, but the author states that he will only take into account one of the main results of this type of research. It concerns the last phase of the genesis of the theory of vector spaces, whose roots lie in the end of the 19th century, but which really began only after 1930, and corresponds to the axiomatization of linear algebra, i.e. a theoretical reconstruction of the methods of solving linear problems, using the concepts and tools of a new axiomatic core theory. These methods worked, but were not explicitly unified; it is important to note that this axiomatization was not, in itself, done to allow mathematicians to solve new problems; rather, it gave them a more universal approach and language to be used in a variety of contexts (functional analysis, quadratic forms, arithmetic, geometry, etc.). The axiomatic approach was not an absolute necessity, except for problems in non-numerable infinite dimension, but it became a universal way of thinking and organizing linear algebra.

One solution might be to forgo teaching the formal theory of vector spaces. However, many people find that it is important for students beginning university mathematics and scientific studies to acquire an idea about the axiomatic algebraic structure of vector space. To achieve this goal, the issue of formalism cannot be avoided. Therefore, it is necessary to induce students to a certain kind of reflection on the use of their previous elements of knowledge and competences in relation to the new formal concepts.

In the so-called obstacle of formalism they distinguish three basic languages used in linear algebra: the 'abstract language' of the general theory, the 'algebraic language' of the theory in Rn and the 'geometric language' of three-dimensional spaces. Another difficulty that seems to be ignored by the teachers is the constant change of notations without alerting the students.

Likewise, in an epistemological study of the connection between geometry and linear algebra, it was found that the need for geometric intuition was very often postulated by textbooks or teachers of linear algebra. However, in reality, the use of geometry was more often superficial.

Research in mathematics education cannot give a miraculous solution to overcome all the difficulties in teaching and learning linear algebra. Several works contain diagnoses of students' difficulties, epistemological and experimental analyses of their teaching, offering local remedies. However, these works lead to new questions, problems and difficulties, which should not be interpreted as a failure.

Improving the teaching and learning of mathematics cannot be a one-size-fits-all remedy. The cognitive processes of mathematics are too complex for a simplistic and idealistic view. It is a deeper understanding of the nature of the concepts and the cognitive difficulties they entail that helps teachers to make their teaching richer and more expert; not in a rigid and dogmatic way, but with flexibility. In this sense, in several countries, research in mathematics education is being influenced by curricular reforms, in a non-formal way, since, in education in North America, everything is determined at the local level, by institution or by teacher. What the group is proposing boils down to a proposal that may or may not be adopted.

3. Other approaches to learning linear algebral

Ana Sierpinska in her article: "On some aspects of students' thinking in linear algebra" focuses on certain aspects of students' reasoning in linear algebra and their difficulties in understanding it. Generally, it is argued that students tend to think practically rather than theoretically, and several examples illustrate how this tendency can negatively affect their reasoning in linear algebra. Specifically, three modes of reasoning in linear algebra can be distinguished, corresponding to their interactions with language, which are: the "visual geometric" language, the "arithmetic" language of vectors and matrices as lists and tables of numbers, and the "structural" language of vector spaces and linear transformations.

A study was conducted at Concordia University, which focused on the patterns of students' formation of mathematical behaviors as they interacted individually with tutors and linear algebra texts. The mathematical content of the tutorial sessions was not specially designed; a guide text was used, which the student read and the tutor probed for understanding, as well as helped the student learn the material.

On the other hand, the Cabri package was used to design a geometric model of two-dimensional vector space. The aim of this study was to try to solve the main problem of minimizing the chances of students developing the symptoms of what is called "the formalism obstacle", thus changing somewhat the notion introduced by Dorier, in which they claim that students' erroneous reasoning in linear algebra derives mainly not only from insufficiency in logic and elementary set theory, but also in the manipulation of algebraic expressions.

In the end, the objective was partially achieved, as some succeeded, but others did not. Drawing on Vygotsky, the author distinguishes between three modes of thinking:

- synthetic-geometric,
- analytical-arithmetic and

- analytical-structural,

whose identification is based on a historical analysis and the study of different "languages" used in the theory of linear algebra itself.

She then describes other experiences that were carried out using Cabri on linearly dependent vectors and linear transformations. By means of the visual results, questions are asked to several students and then these answers are analyzed by the author, to which she concludes: "the answers given by the students are descriptions that cannot be understood outside of what they could see on the screen"[3].

Then, more experiences of tests carried out with Cabri are shown, where it is concluded that many could not generalize the properties proposed by the researchers, since they remained in what they saw on the computer screen. Others believed that the objective was to study geometry and did not relate it to linear algebra. The author concludes that, in this research on the teaching of linear algebra, it was found that no matter how one tries to approach the content, the students' difficulties seemed to persist.

Ed Dubinsky, American mathematics professor and creator of the so-called APOE theory. (Action-Process-Object-Scheme), who has made numerous contributions to mathematics education, in his article "Some Thoughts on a First Course in Linear Algebra at the College Level" defines two purposes: the first is to make a sharp criticism of the recommendations made by the LACSG group on the teaching and learning of linear algebra and the other is to propose an alternative approach based on the APOE theory.

In the criticisms made of David Carlson's article on the recommendations given by the LACSG group for teaching linear algebra, Dubinsky first states that students' difficulties are not only toward vector space concepts, but that students also have difficulty learning the concepts of linear applications. Furthermore, he says that the reasons Carlson gives are very general and do not clearly point to how they can be overcome.

The author agrees with the position of the LACSG group that the teaching-learning processes should be more dynamic, he also agrees that there should be applications and that students should be stimulated with problems, but notes that the group does not say how to do it. That is why he proposes a constructivist approach, as he argues that "the pedagogical approach in most linear algebra courses is, and has been for a long time, to tell students about the mathematics and show them how it works. The closest students come to playing an active role is when they work on problems. But this does not work very well because, as a profession, the author believes that teachers succumb to the demand of students who first show them how to do a certain type of problem and then ask them to solve many instances of this same problem. To summarize, students do not understand these concepts because they never have the opportunity to build their own concepts.

He proposes the APOE theory, as a teaching and learning process for linear algebra, which he presents as follows.

Action. It is a transformation, it is considered an action when it is a reaction to externally perceived stimuli. These stimuli, and the reaction, can be physical or mental. An action can be either a single-step response or a multi-step sequence of responses, i.e. when a student performs a series of steps without much sense or, as is often said, mechanically, as for example when the student learns to reduce a matrix by applying an algorithm.

Process. It occurs when an action is repeated and the individual reflects on it, can come to be perceived as a part of the individual and that he or she can establish control over it; that is, it is an internalization of the action. Once an individual has constructed a process, several things are possible. For example, two or more processes can be connected. Perhaps the most important point is that when it comes to a process, an individual can think of a transformation. For example, in linear algebra when a student is able to express the steps necessary to determine without verification that the sum of two elements of a set is or is not in the set, then the subject can be said to have internalized the actions in process.

Object. When an individual reflects on the operations applied to a particular process, and realizes the process as a whole and the transformations (whether actions or processes) that can act on it, then he can construct his transformations and will be thinking of this process as an object. For example, in linear algebra, when a student is able to demonstrate when a given set is a vector subspace.

[3] Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra in DORIER J.-L. (ed.), *The Teaching of Linear Algebra in Question*, pp. 209-246. @2000 Kluwer Academic Publishers.

Professor Chargoy has also conducted research on the learning process of linear algebra and in her article: "Synthetic and analytical modes of thought: the case of the basis of a vector space" deals with the analysis of students' thinking when solving problems on bases of vector spaces and the analysis of six linear algebra guide texts on the presentation of these concepts to the reader. The researcher says that, in her teaching experience, students have difficulties in learning linear algebra because the content of the subject contains a large number of symbols, definitions, properties, concepts and theorems with a high degree of abstraction.

She develops her work on vector spaces, which she considers as the fundamental concept of linear algebra and the bases of spaces or subspaces as the skeleton that supports the development of linear algebra. With respect to her experience and the literature of linear algebra textbooks, she says that algorithmic and arithmetic procedures prevail more than geometric and conceptual ones.

The research has as its theoretical framework Anna Sierpinska's theory of modes of thought, which identifies three modes of mathematical thought: the synthetic-geometric, the analytic-arithmetic and the analytic-structural. "The main difference between the "synthetic" and "analytic" modes is that, in the synthetic mode, objects are given directly to be described by the mind, in a natural way, while in the analytic mode these objects are given indirectly. In fact, they are constructed only by the definition of the properties of the element".

In this research, engineering and mathematics undergraduate students are given problems on bases of vector spaces to be solved in groups and individually by them, then interviews are conducted on how they solved the problems to analyze the type of thinking and the difficulties that arise in each case.

The problems posed deal with bases of subspaces in R^3 and R^2 and in this context the author concludes that students are confused between the number of elements of a basis and the dimension of these elements. For example, given three linearly independent two-by-two vectors in R^2, students believed that they could generate a basis for R^3. The author also concludes that most students usually work in the analytic-arithmetic mode of thinking.

Finally, we have in Colombia a very important research on the teaching and learning of linear algebra in Colombia made by García (2017) where a mathematical model and a methodological process for the teaching of this subject based on plausible reasoning and the use of technology is proposed.

## IV. Conclusions and Recommendations

The studies carried out on the teaching and learning of linear algebra provide the fundamental scientific elements that support the problem of this proposal. Most of the authors consulted will constitute the theoretical support to be able to propose a methodology for the teaching and learning of linear algebra, as indicated at each moment.

On the other hand, the knowledge of the studies that have been carried out on the learning problems of linear algebra was not only necessary, but also guarantees an original approach to the theoretical and practical contribution to be made with this work and with them novel results for mathematics education.

In addition, the study reflects the importance of linear algebra in engineering careers, since, at the beginning of the 90's, both in the United States and in Europe and other parts of the world, different academic groups began to organize and form to study the problem of teaching and learning this discipline. Each of these groups made recommendations to be taken into account for the teaching of the subject, since there is no formula or recipe to solve the problem, but there is a field to investigate, propose and improve its teaching and learning and thus make a contribution to achieve a mathematics education of excellence in Colombia and the world.

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