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Exploring Cubic Vertex-Transitive Graphs: A Comprehensive Study Up To 1280 Vertices

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Abstract

This paper presents a comprehensive study of cubic vertex-transitive graphs with an order of up to 1280 vertices. Additionally, tetravalent arc-transitive graphs with an order of up to 640 vertices are also determined. The investigation combines novel theoretical insights with rigorous computer calculations to identify and classify these graph structures. The results contribute to a deeper understanding of the properties and patterns exhibited by cubic vertex-transitive graphs, offering valuable insights for further research in this field.

Keywords: Cubic graphs; Vertex-transitive graphs; Tetravalent graphs; Combinatorics.

1. Introduction:

There has already been a lot of studying done on vertex-transitive structures, presenting intriguing questions that remain open and actively pursued **(Kutnar et al 2012).** Throughout this paper, we focus on finite and simple graphs, they have no dual edges, are unfocused, and lack loops. If an automorphism that defines a graph operates intermittently on its vertex set, the graph's structure is said to be edge-transitive. Considering connectedness, which is preserved through pairwise isomorphic connected components, allows us to assume connectedness without loss of generality.

One important finding reveals that a corner-transitive circuit can only be regular and have the exact same values for every vertex. The graph is unavoidably cycle-like for valences of no more than two and an order of less than three. Therefore, cubic visualizations, which are symmetrical triangles possessing a valiancy of 3, are the most recent non-trivial instance. As a result, many investigations into vertex-transitive graphs initially focus on the cubic case. However, even in this scenario, numerous questions remain challenging, given our current understanding of cubic vertex-transitive graphs.

Various authors have made attempts to explore cubic vertex-transitive graphs of specific types, considering factors such as order and simple factorization. Previous works, such as those by **(Jin, & Tan, 2022)** provided valuable insights by enumerating and constructing tables of cubic vertex-transitive graphs up to certain orders. Nevertheless, to the best of our knowledge, the best available results up to now were limited to orders not exceeding 94.

In this paper, we take a step further by presenting a comprehensive census of all the 1280-ordered cubic vertex-transitive diagrams. We found that isolation is possible, a total of 111360 such graphs, though we do not provide an exhaustive list in this paper **(Potocnik, et al 2021).** Instead, we offer an overview of the overall data in methods, with a complete list available in Magma code online **(Zimmermann, et al 2018)** To achieve these results, we employ various methods and techniques, which we outline in the following sections while establishing essential terminology and presenting basic results.

2.Research Methodology

The primary focus of this paper is to determine and classify all cubic G-vertex-transitive graphs with an order of up to 1280 vertices. The investigation involves three distinct cases, each with its unique challenges and methodologies.

1. Case m=1 "(Arc-Transitive Graphs)": In this scenario, the parameter "m" representing Each vertex stabilizer's orbital count G_v in its action on the neighbourhood Γ(v) is equal to 1, indicating an arc-transitive graph. The arc-transitive case is relatively straightforward, and Conder previously built a database that includes all cubed arc-transitive diagrams that exceeds a certain number of 10000 using a computer-assisted approach. The method relies on Tutte's theorem, which bounds the order of the vertex-stabilizer group $|G| \leqslant 48|V(\Gamma)|$.. Consequently, the method efficiently identifies all such graphs by exploiting the specific group structure and applying the Low Index Normal Subgroups algorithm in Magma.
2. Case m=3 (Isolated Vertex): When m=3, the vertex-stabilizer G_v fixes the neighbors of v pointwise, resulting in G_v=1 due to connectedness. This lack of structure in the vertex-stabilizer poses challenges in applying the previously successful method used for arc-transitive graphs. However, the fact that G_v=1 implies $|G|\leqslant|V(\Gamma)|\leqslant 1280$, which enables Volcano to access the Small Groups repository. The method involves using specific techniques to restrict the search space effectively, enabling the computation of all possible groups G and their corresponding graphs Γ.
3. Case m=2 (Intermediate Case): The case where m=2 represents the most challenging scenario. The vertex-stabilizer can have arbitrarily large order, making the direct application of previous methods infeasible. However, an auxiliary graph construction technique is introduced, leading to the creation of a tetravalent and G-arc-transitive graph with half the order of Γ. Importantly, it is demonstrated that this construction can be reversed. Thus, the problem of finding all cubic "G-vertex"-transitive graphs with "m=2" reduces to identifying all tetravalent arc-transitive graphs with an order of at most "n/2". Fortunately, a recent result **(Potocnik, et al 2021).** establishes that,

apart from specific exceptions, the automorphism group's order in a quadratic measure of the diagram's order bounds a hexagonal arc-transitive structure upper. By using this discovery, a technique like the one used with the hexagonal arc-transitive situation may be used to quickly locate the pertinent structures.

This paper combines theoretical insights with computer calculations to tackle the challenging task of determining complete vertex-transitive triangular diagrams with an order of up to 1280. The research methodology involves specialized techniques, exploiting group properties, and employing the computational capabilities of Magma and Small Groups databases to achieve the desired classifications. It encompasses three distinct cases ("m=1, m=2, and m=3"), with each case requiring tailored approaches to address the specific complexities presented.

Table 1.1 Number of cubic vertex-transitive graphs of order at most 1280.

	$m = 1$	$m = 2$	$m = 3$	Total
Cayley	386	11853	97687	109926
Non-Cayley	96	1338	0	1434
Total	482	13191	97687	111360

2.1Data about the graphs

Our report has plenty of graphics to go through each one separately in this essay. via the internet (Potocnik, et al. 2021) is where you can get the whole list of graphics that make up the Feldspar code. Whilst doing so, we only provide a review of them along with some general figures.

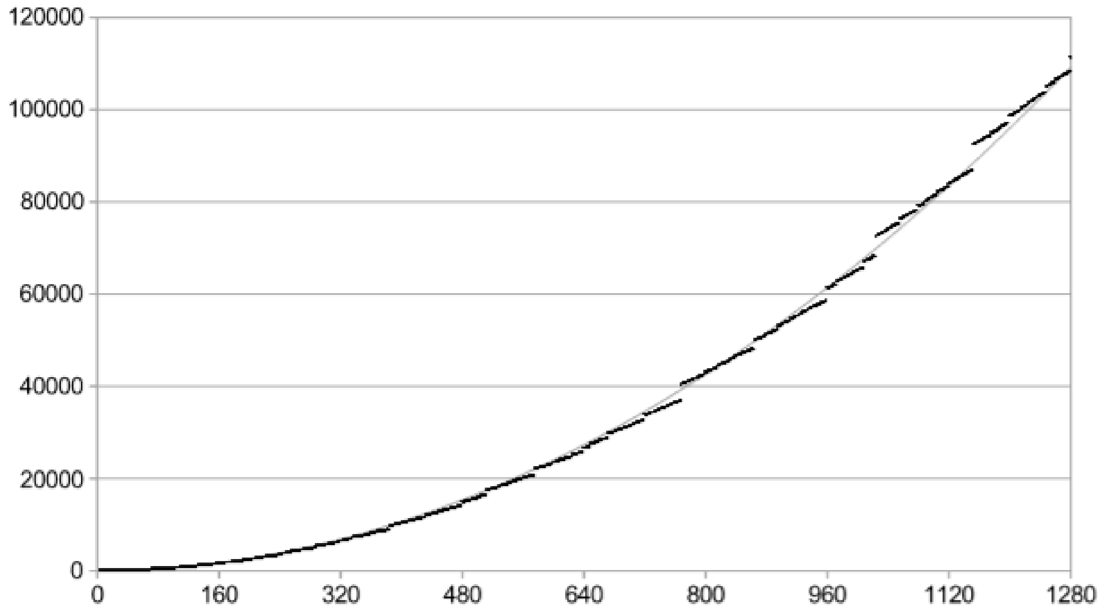

Fig. 1 demonstrates the quantity of cubed vertex-transitive circuits of at most n with respect to n (The black points on the map are them. The visualization of the function's results is overlaid on the data in question $n \mapsto n^2/15$ It is shown as a narrow, light-gray curve and looks like to have very similar to the gamut under consideration. This is a tendency that won't likely persist. In an article that will be published soon, **(Potocnik, et al 2021).** we prove that if f(n) is the number of cubic vertex-transitive graphs of order at most n, then $\log(f(n)) \in \Theta((\log\ n)^2)$. In this sense, high quantities happen c_1 and c_2 It happens for any sufficient size n,

"$c_1(\log\ n)^2 \leqslant \log(f(n)) \leqslant c_2(\log\ n)^2$".

Also take notice that despite the fact that a fluid function may reasonably mimic some of the points displayed in Fig. 1.1, certain bigger leaps can still be clearly recognized. where n has several minor significant variables. (For example, at 768,1024 and 1152).

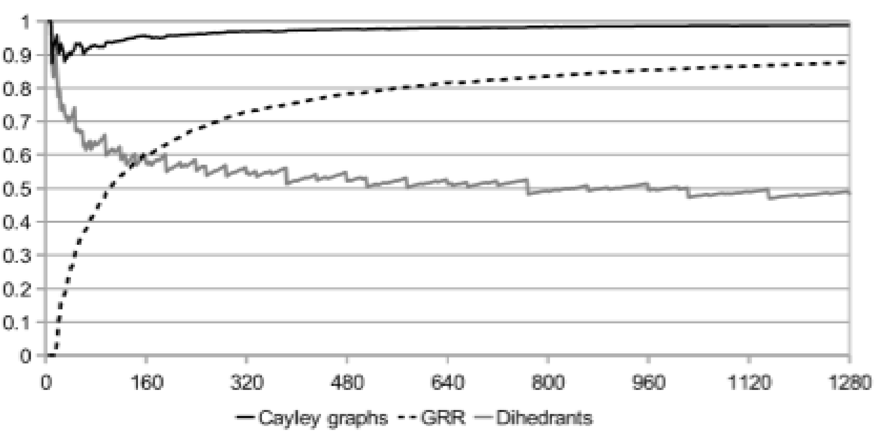

Fig. 1.2. Proportion of graphs of different types of order at most n.

3. Results and Discussion:

The paper presents a comprehensive analysis of cubic G-vertex-transitive grap1152) h an order of up to 1280 vertices, classified based on whether they are Cayley graphs and the parameter to consider is "m," which denotes how many arc-orbits make the overall automorphism category. The following key findings are highlighted:

1. Distribution of Graph Types: Among the cubic vertex-transitive graphs in the given range, the majority are Cayley graphs, and within this group, a significant portion are graphical regular representations (GRRs). This trend is observed in Table 1.1 and Figure 1.2, where the proportion of GRRs tends to approach 1. This supports the conjecture that most vertex-transitive graphs are likely to be Cayley graphs, particularly within the cubic case.
2. Dihedrants: The proportion of dihedrants initially appears substantial but stabilizes at approximately half within the range of prices under consideration. nevertheless, the percentage of dihedral angles tends to 0, If the total quantity of cubes dihedrants increases, it does so most polynomially n, As opposed to any polynomials functioning, the entire number of vertex-transitive topologies increases more quickly n.
3. Hamilton Cycles: All graphs in the census, except for well-known exceptions, contain at least one Hamilton cycle, which indicates the existence of a continuous cycle visiting each vertex once.
4. Cage Problem: The girth values of the graphs in the census are examined to potentially yield new extremal examples for the cage problem. The values of $n_{vt}(3, g)$, $n_{cay}(3, g)$, and $n_{at}(3, g)$ for $g \leqslant 16$ are provided in Table 1.2, and previously unknown values are reported.
5. Degree Diameter Problem: The diameter of each graph in the census is computed to assess them against established extreme cases for the extent of girth issue. The values of $m_{vt}(3, d)$, $m_{cay}(3, d)$ and $m_{at}(3, d)$ for $d \leqslant 8$ and bounds on these when $9 \leqslant d \leqslant 11$ are presented in Table 1.3. Several of these values and bounds are new, contributing to the understanding of the extreme values of the degree and diameter in cubic vertex-transitive graphs.

The research provides a comprehensive analysis of cubic G-vertex-transitive graphs and sheds light on the distribution of graph types, revealing intriguing trends and supporting conjectures about the nature of vertex-transitive graphs, especially in the cubic case. Additionally, the investigation explores the presence of Hamilton cycles and provides new insights into extreme cases of the amount of radius issue and corresponding cage situation in cubic vertex-transitive graphs.

Table 1.2 lowest quantities for any specific circumference.

g	$n_{vI}(3, g)$	$n_{cry}(3, g)$	$n_{ar}(3, g)$
3	4	4	4
4	6	6	6
5	10	50	10
6	14	14	14
7	26	30	28
8	30	42	30
9	60	60	60
10	80	96	80
11	192	192	506
12	162	162	162
13	272	272	2600
14	406	406	448
15	620	864	620
16	1008	1008	1008

Table 1.3 Largest orders for given diameter.

d	$m_{vt}(3, d)$	$m_{cay}(3, d)$	$m_{ar}(3, d)$
2	10	8	10
3	14	14	14
4	30	24	30
5	60	60	60
6	82	72	64
7	168	168	168

8	300	300	234
9	⩾ 546	⩾ 506	364
10	⩾ 1250	⩾ 882	1250
11	⩾ 1250	⩾ 1250	1200

4. Conclusion

Data presented in Tables 1.2 and 1.3, we draw the following conclusions regarding cubic G-vertex-transitive graphs with an order of up to 1280 vertices:

1. Girth: The smallest orders for given girth values are provided for three different types of graphs - vertex-transitive. Notably, for girth values up to 8, the orders are relatively small and consistent among the three graph types. However, for higher girth values (9 and above), the Cayley graphs and arc-transitive graphs show a significant increase in order, with some values being substantially larger than the corresponding vertex-transitive graphs.
2. Diameter: The largest orders for given diameter values (from d=2 to d=11) are reported for the three graph types. The diameter values up to 7 exhibit comparable orders among all graph types. However, for larger diameter values (8 and above), the Cayley graphs and arc-transitive graphs demonstrate higher orders than the vertex-transitive graphs.
3. Comparison: The data suggests that the Cayley and arc-transitive graphs tend to have higher orders than the vertex-transitive graphs for both girth and diameter values. This observation aligns with the overall trend identified in the study, where Cayley trees make up a substantial proportion of the total number of cubed vertex-transitive diagrams although the most of them are graphical regular representations (GRRs).
4. Extreme Values: The results provide insights into extremal examples for the cage problem and degree diameter problem in cubic vertex-transitive graphs. Notably, some of the orders for specific girth and diameter values (e.g., g=11, d=10, and d=11) are significantly larger than others, indicating interesting and unique structural properties for these specific graphs.

In conclusion, the study of cubic G-vertex-transitive graphs with an order of up to 1280 vertices reveals intriguing patterns in the distribution of graph types based on girth and diameter values. The dominance of Cayley graphs, particularly GRRs, and the emergence of extreme values for specific graph types offer valuable insights into the nature and complexity of vertex-transitive graphs in the cubic case. The results contribute to the ongoing research in graph theory and provide a deeper understanding of the relationships between graph types and their structural properties.

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