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State of the art on the characterization of insight types in mathematical problem solving

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ABSTRACT- This article presents a selection of papers that have been considered relevant to this research. These are related to divergent and convergent thinking, insight and creativity in mathematical problem solving, without them representing all the material that may exist regarding these topics, and that have been published from the second half of the last century to date. However, a careful selection of diverse authors with very representative and high quality results should allow us to orient the research towards a theoretical framework and a successful methodology of work.

KEYWORDS: State of the art, Divergent thinking, Insight.

I. INTRODUCTION

The vast majority of careers related to the degree in mathematics and mathematics, such as engineering, physics, chemistry, administration, among others, present mathematical analysis content in different courses under the name of "calculus" or "mathematics" where inequalities and curve analysis occupy topics or a more or less significant number of hours. These contents have been characterized by difficulties in their learning and in the development of students' skills, where they are often presented in a traditional way with evaluative demands limited to the resolution of standard exercises. In the end it turns out that "good results obtained" by the students themselves make them believe that they have the necessary skills, their parents who observe the grades obtained by their children and what is worse - the teacher himself - believing that he is doing the right thing, Garcia p., m. (2012). This state makes us lose the essence of being able to enhance the different thoughts in mathematics, which are ultimately those that validly contribute to true learning.

When an exercise is transformed into a problem to be presented to the students, it is possible that an initial blockage is being created in them, not knowing where to start, but motivating them to look for a solution, making use of their preconceptions and concepts acquired during their learning, then the students will be contributing to the development of creative thinking and a feeling of respect for the teacher.

II. METHODOLOGY

An analysis of research on divergent thinking and the different types of insight in mathematical problem solving and the development of mathematical thinking in students in their fifth semester of undergraduate studies in mathematics at a university in Bogota, Colombia was carried out:

a) **Methodology applied.** This study is developed under a qualitative approach, where a documentary review of the state of the art in some databases is carried out.

b) **Procedure.** In order to carry out this work, a search for research that dealt with the different types of insight that students have when solving mathematical problems and in addition to divergent thinking was carried out, these researches were classified to be analyzed by categories and areas of interest.

For the analysis of scientific articles, the study focused on their activity and impact. In the activity indicators, the current state of science was visualized and within this, the number of publications and their productivity were analyzed. For the impact indicators, the number of citations of the research was taken into account, which gives a characterization of the importance of the document and the recognition given to it by the researchers.

III. RESULTS AND DISCUSSION

1. About insight according to the psychological sciences

1.1 Creativity and insight [1]

Martín, C. (1999) highlights the most classical contributions of Gestalt psychology, a German term that alludes to creativity.

When solving problems, some people suddenly feel that they know the answers, without being able to explain how they have achieved it (Metcalfe, 1986; Wallas, 1926). The above experience of the so-called "Aha" has formed the basis of some definitions of *insight* according to (Duncker, 1945; Kohler, 1956; Maier, 1930). Some contributions have shown and suggested that *insight-type* problem solving is contrary to routine and less sudden forms of problem solving according to (Gruber, 1979; Nickles, 1978).

Research on *insight* can be found in the studies of Gestalt psychology (Dunker, 1945, Maier, 1930, Wertheimer, 1959) whose authors believed that *insight* is a process that could be qualitatively differentiated from other types of mental processes and has the following characteristics:

1. It corresponds to a sudden restructuring of a problem that is accompanied by the sensation of "unconscious leaps" in thought.
2. It is an accelerated mental process.
3. It is a kind of short circuit in the normal reasoning process.

Hadamard (1949) offers an explanation of the phenomenon of *insight*, in which he identifies four stages that describe scientific *insight* very well: preparation, incubation, illumination, and verification. These characteristics constitute a set of empirical generalizations related to *insight*.

The first stage involves a major effort in trying to solve a given problem; for some problems that attempt may lead directly to a solution. But for more difficult problems, the person may give up or turn away.

Abandonment allows entering the next stage of incubation, in which at a conscious level the person is processing other things. Depending on the situation, this second stage can last from seconds to years, but the person experiencing the *insight* proposes the solution in the most important stage which is enlightenment, which must occur unexpectedly and quickly, known as the "Aha" experience.

This leap can lead to false insights so the fourth and final stage, verification, is the one to go over all the details.

The following table summarizes the four stages of insight according to Hadamard

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[1] Martín, C. (1999). Creatividad e insight. *Revista de altas capacidades.* ISSN 1136-8136, Nº.7, pp. 63-84 https://dialnet.unirioja.es/servlet/articulo?codigo=2476241.

Table 1. Stages through which insight passes according to Hadamard (1949)[2].

STAGES	FEATURES	SUBJECT
1. PREPARATION	• Great effort • Consciousthought • Generaterelevant ideas	• You can give up if the task is complex.
2. INCUBATION	• It can last seconds or years. • Unconsciousprocess.	• Other information can be processed.
3. ILLUMINATION (insight)	• It's "Aha" or "eureka" time. • There can be false "Ahas".	• Notifies and makes explicit what was given in the incubation.
4. VERIFICATION	• Reviewdetails • It can be shown that there is no insight.	• Resolveconsciously.

It is considered that this work presents aspects that will be of great utility for the present investigation, when trying to describe the possible insights that can emerge in the course of solution of problems raised in the classroom.

1.2. Psychology of science and creativity [3]

This research is based on the study of scientific creativity in the framework of the psychology of science, it analyzes the previous origins since this discipline of the meta-sciences was consolidated as the form of study of a scientific epistemology. Four research methods used in the study of creativity are reviewed: psychometric, experimental, historiometric and case study. It is defined as a subdiscipline that emerges between two fields such as psychology and mathematics. Feist stated that: *"The psychology of science is the empirical study of the biological, cognitive, evolutionary, social and personality influences on scientific thought and behavior"[4].*

Romo (2009) understands *"creativity as a way of thinking whose result is a product that has both novelty and value"[5].* She points out that the case study can be of great interest and useful, as pointed out by Gruber[6] who published a psychological study of Darwin's scientific creativity. This study has been the starting point of some researches about creation, which later became part of the twelve cases presented in another work by Gardner[7].

Gruber discusses the missteps in the lives of great scientists, as well as the heuristic role of images that condense and organize information.

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[2]Martín, C. (1999). Creatividad e insight.*Revista de altas capacidades.* ISSN 1136-8136, Nº. 7, 1999, pp. 63-84 http://dialnet.unirioja.es/servlet/articulo?codigo =2476241

[3] Romo, M. (2007). Psicología de la ciencia y creatividad. *Revista creatividad y sociedad.* Nº 10, pp. 7-31, Marzo de 2007, disponible en http://www.creatividadysociedad.com/articulos/14/Creatividad

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[4] Feist, G. (2006). *The Psychology of Science and the Origins of the Scientific Mind.* Yale UniversityPress, p.4.

[5]Romo, M. (2009). *Psicología de la creatividad.* Barcelona: Paidós (segunda impresión Marzo de 2012).

[6]Gruber, H. E. (1974). Darwin sobre el hombre. Un estudio psicológico de la creatividad científica. Madrid. Alianza, 1981.

[7] Gardner, H. (1993). *Creative minds. An anatomy of creativity*. (trad. cast.: *mentes creativas*. Barcelona, Paidós, 1995).

It is considered of great importance for this research to implement the case study that in some way will allow to follow up or observe students who may be presented with some type of *insight* inside or outside the classroom.

1.3. Studying insight problem solving with neuroscientific methods[8]

This research studies methodological aspects of brain research when *insight* occurs, describing the processes that occur during problem solving as short-lived moments of exceptional thinking.*Insight* requires a restructuring of the problem situation that is relatively rare and difficult to obtain in the laboratory. One way to address this problem is to catalyze these restructuring processes using solution cues, which allows multiple moments of *insight*, and their onset times, which are required for event-related designs in functional magnetic resonance imaging (FMRI) and electroencephalogram (EEG), to reliably record the activity associated with the restructuring component of *insight*. The research hypothesis holds that, during the restructuring process, new associations are formed between existing knowledge nodes.

According to the authors, *insight* can occur in the process of problem solving, in which the same parts of the brain are not activated in the restructuring, since it is a dynamic cognitive process that requires a broad reorganization of the representation of the problem.

1.4. Intuition, Incubation, and Insight: Implicit Cognition in Problem Solving[9]

The authors describe the stages of thinking according to what Wallas (1926) proposed, where intuition, incubation, and insight are seen as "an individual achievement of thought" (p. 79). He breaksdownthestagespresent in problemsolving:

- The *preparation* stage consists of the accumulation of knowledge and mastery of the logical rules that govern the particular field in which the problem resides. It also involves the adoption of a definite attitude toward the problem, including the realization that there is a problem to be solved and the analysis of the problem itself. Sometimes the problem is solved at that point. Usually, this occurs in the solution of routine problems, in which the application of an algorithm, allows to reach the correct solution.
- The *incubation* stage, once the problem is understood, it is time to internalize it unconsciously. In this stage the person abandons the problem, abandonment that can last from hours to months, apparently stopping the creative process.
- In the *enlightenment stage*, the sudden idea comes, the *flash*, what the person was looking for so insistently may come at the least unexpected moments when the person may be doing different activities.
- The *verification* stage is the space in which the solution is evaluated and rectified, sometimes the person goes back to the incubation stage, to find a new solution.

In addition, the authors describe that, in the solution of problems by insight, according to Bowers (1990, 1995), the automatic and unconscious activation of the knowledge that each individual possesses is present.

These aspects can be useful in the description of the possible types of insight that can occur in the process of solving mathematical problems inside or outside the classroom, also mentioned by Hadamard (1949).

1.5. The Language Instinct: How the Mind Creates Language[10]

Steven Arthur Pinker is a Canadian experimental psychologist, cognitive scientist, linguist and writer. He is a professor at Harvard College and holds the Johnstone Family Professorship in the Department of Psychology at Harvard University. His academic specialties include perception and language development in children.

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[8] Luo, J. Knoblich, G. (2007). Studying insight problem solving with neuroscientific methods. *ScienceDirect.* pp. 77-86. Available online at www.sciencedirect.com.

[9]Dorfman, J. Shames, V. Kihlstrom, J. (1996). Intuition, incubation, and insight: Implicit cognition in problem solving. Underwood, Geoffrey D. M. (Ed), (1996). *Implicit cognition*. (pp. 257-296). New York, NY, US: Oxford University Press.

[10] Pinker, S. (1994). *The Language Instinct: How the Mind Creates Language*. p 437. (material sugerido por la Dra. ConstanceBohanon, consultora de este trabajo)

With respect to the teaching of mathematics and physics, Pinker states that the processes of learning mathematics can be facilitated by means of two of the seventeen mental modules that he has postulated, those of **intuitive mechanics** and **number**, working first on the construction of concepts and rules, and then introducing the graphics and symbolic systems that represent numbers and relationships with written equations. He comments that, mistakenly, the traditional educational system begins with symbolic and written representations in which memorization and abstraction are used and achieves little understanding in students.

Another research consulted was that of Sharma (1979) who describes the reasoning process of the human being through the right hemisphere and left hemisphere of the brain, pointing out that there are two profiles for learning geometry which he calls visual and verbal in nature.

1.6. Aha!: The effect and affect of mathematical discovery on undergraduate mathematics students[11]

Liljedahl (2004) in his work answers the question What impact would the experience of Aha! (*insight*) would have on a group of students about to finish their degree? A qualitative methodology based on observation, monitoring and interview of a sample of 76 undergraduate mathematics students at Simon Fraser University, enrolled in the course Fundamentals of Mathematics for Teachers and whose main objective was to develop the understanding of mathematics at the elementary level, was used. It lasted thirteen weeks with an intensity of four hours per week and followed up and analysed the students who chose to describe their experience of the so-called Aha!

The data for this study came from the final semester project, one of the options was to write about a mathematical Aha! that they had experienced during their participation in the course.

The project was worth 10% of their final grade and they had four weeks to work on it. In one part of one of the projects submitted by a student he refers to an Aha! experience in the following terms: *"I had been working on the problem for a long time without any progress. Then suddenly I knew the solution, I understood, it all made sense. It seemed like it had just clicked! "* [12]

The purpose of the projects was to reflect on the Aha! experience, and to explore what was learned in each case.

According to the responses of the participants of all the projects, four factors stand out in relation to the experience of the Aha!

1. Anxiety

Some students expressed some uneasiness about the mathematics course they were going to take, in addition to their changing feelings towards it, manifested in terms of antipathy, fear, dread, and traumatic memories.

2. Pleasure

The Aha! experience, in the vast majority of students produced a positive change, which in some way contributed to the change of beliefs and attitudes towards mathematics, and created a greater interest in finding a solution.

3. Change of beliefs

Some students focus on their own conceptions around their abilities and how they do mathematics, and how these beliefs can change, as seen in the following expression from one student, *"I used to think math was all about the right answer, but now I am more aware of the value of the process."* [13]

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[11] Liljedahl, P. (2004). AHA: The effect and affect of mathematical discovery on undergraduate mathematics students. A paper presented in TSG3 at ICME-10. Available online at http://www.icme-organisers.dk/tsg03/TSG3_Liljedahl.pdf.

[12] Liljedahl, P. (2004). AHA!: The effect and affect of mathematical discovery on undergraduate mathematics students. A paper presented in TSG3 at ICME-10. Available online at http://www.icme-organisers.dk/tsg03/TSG3_Liljedahl.pdf.

[13] Ibid.

4. Attitude change

The author considers that attitudes are the manifestations of beliefs, sometimes it was difficult to distinguish between the two, since any expression of a change in attitude, had a perceptible change in the beliefs associated with it. In this sense, he highlights the experiences of two students:

"I have a better attitude; I am now more optimistic. This is helpful in learning the processes of a thought, which can be impeded by an attitude of not feeling challenged." [14]

"Also, I like mathematics now. I feel like this achievement spurred to have more successes. I have now expanded my expectations in math." [15]

In the conclusion of this research, they show that the experience of illumination (Aha!) is not only characteristic of great mathematicians. Its power is based on transforming attitudes and beliefs in learning in mathematics, with which it is totally agreed, since if this experience is related to what Fauconnier and Turner (1998 and 2002) referred to, it is part of what are called the previous mental spaces described above, and which shows insight as a mediator towards new mental spaces. However, nothing is mentioned in this research regarding different types of insight that might emerge within convergent and/or divergent mathematical thinking. Many of the methodological ideas of this article have been present in the proposal of the two courses that were developed in the Bachelor's Degree in Mathematics of the UAN, fundamentally in the evaluative aspects and on the importance of propitiating a pleasant academic climate in the accompaniment on the part of the teacher. (See Annex 1.)

2. On Divergent and Convergent Thinking

2.1. Divergence and convergence of mental forces of children in open and closed mathematical problems[16]

Sak, U. and Maker, C. (2005) investigated divergent and convergent thinking relationships based on fluency, originality, flexibility, and elaboration within students' mathematical activity. A section of a performance-based assessment was used to assess 867 students in grades 1-6 in their mathematical proficiency. Statistically, the authors found significant correlations between the components of divergent and convergent thinking, validating the "*Problem Continuum"* matrix *(Schiever and Maker, 1991, 1997)*, where correlations between problem types can vary according to the proximity of the types to each other. The authors make reference that when solving problems one should abandon the idea of thinking as one usually does, and try to think in a non-habitual way, this way of thinking is called flexible thinking.

As for divergent thinking, they are in agreement with what Runco (1990) defines divergent thinking as the generation and application of many different ideas to solve a given problem, being considered as a good creative predictor. For Guilford (1967), the main objective of divergent thinking is in the quantity and quality of the ideas generated by the solver. He also refers to Cropley (1992, 1999) who elaborated the distinction and relationship between divergent and convergent thinking. According to the latter, early researchers in creativity tend to separate the two types of thinking and consider them as functions in different forms of giftedness. As for the students' part he argues that gifted students are possessors of both convergent and divergent thinking; moreover, very varied.

These references can be useful for the design and selection and/or modification of the types of closed problems proposed in the different activities.

2.2. Convergent/divergent cognitive styles and mathematical problem solving[17]

This author posits that the approaches used by students in solving problems vary significantly and these variations are due to the different methods that require conceptual understanding and visual forms of

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[14]Liljedahl, P. (2004). AHA!: The effect and affect of mathematical discovery on undergraduate mathematics students. A paper presented in TSG3 at ICME-10. Available online at http://www.icme-organisers.dk/tsg03/TSG3_Liljedahl.pdf.

[15] Ibid.

[16]Sak, U., Maker, C. (2005). Divergence and convergence of mental forces of children in open and closed mathematical problems. *International Education Journal*, 2005, 6(2), pp. 252-260. http://iej.cjb.net.

[17]Alamolhodaei, H. (1997). Convergent/divergent cognitive styles and mathematical problem solving. *Journal of science and mathematics education* in s.e. Asia vol. xxiv, no. 2.

solution. The main objective advanced by the author was to find the difficulties that students had in solving the problems in terms of comparing the two types of thinking, convergent and divergent, in solving mathematical problems of graphical computation. This study was conducted on third year mathematics majors of students at Ferdowsi University of Mashhad in northeastern Iran with a sample of 93 students who were taking mathematics courses as part of their majors.

The data analysis procedure was done mainly through the use of mean score; the study analyzedone-way variance which is applied to compare the significant differences of two or more means by means of F distribution. This instrument was used to determine the differences in students' performance in solving graphical problems, this was classified into convergent, intermediate and divergent. The results of the activities were presented in the following tables:

Results of theresearch[18]

Table 2
Statistical information of (Convergent/Divergent) tests

Group	Mean Score	SD	Maximum Score	Minimum Score
N=93	54.53	11.30	32.00	92.00

Table 2
The distribution of cognitive styles over the sample

Group	Convergent	Intermediate	Divergent
N=93	48.6%	22.58%	36.55%

The analysis of the results showed a significant correlation between the types of thinking present in the resolution of mathematical problems where visual forms are involved in calculus topics. They suggest the planning of activities that encourage both convergent and divergent mathematical thinking; however, it is considered that this issue should not be easy to plan a priori, since the same problem can provoke convergent thinking in one subject, divergent in another and in another the situation of not finding any solution.

IV. CONCLUSIONS AND RECOMMENDATIONS

When analyzing the scope of the results of the previous works, it can be concluded that they constitute a good base to reach a very good investigation on the objectives for the characterization of the possible insight that can be appreciated and that arise in the school activity with students who are challenged to solve mathematical problems posed in classes.

Most of the methodologies implemented in previous research are qualitative, which suggests that the implementation of a case study for the detection of characteristics of this type of cognitive phenomenon (insight), which presumably can be observed in the processes of solving problems posed in the classroom is a wise decision.

The results of this work allow us to adjust a theoretical framework. In addition, it should suggest a flow diagram that connects problem solving with insight occurrences, as well as, clarify the relationship of these with convergent and divergent thinking.

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