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Optimal Policies For Complementary And Substitute Items In Uncertain Environment Under Imperfect Production Process

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Abstract

In this paper, the optimal production for an inventory control system of multi-items where items are either complementary and/or substitute is formulated with an uncertain resource constraint. Here, the production function is unknown and considered as a control variable. Also, the defective rates of the items are reliability dependent. The demand is stock dependent, shortages are not allowed. Unit production cost is reliability parameter and production rate dependent. It also includes the cost due to the environment protection measures. Set up cost is also reliability parameter dependent. The total profit, which consists of the sales proceeds, production cost, inventory holding cost and set up cost is formulated as a Pontryagin's Optimal Control problem and evaluated using Taylor's theorem, generalized reduced gradient technique and optimal control theory satisfying the Generalized Legendre conditions. Here, the inventory costs are fuzzy rough in nature and the resource constraint is imprecise. Fuzzy rough expectation is introduced and taken over the inventory costs and the constraint. The model is formulated in general form for n-items and in particular, is illustrated with three items for some numerical data. The optimum results are presented both in tabular form and graphically.

Keywords: Multi-item inventory; Pontryagin's maximum principle; Complementary and substitute item; Fuzzy rough expectation; Reliability.

Introduction

Different types of uncertainty such as randomness (Ishii and Konno 1998), fuzziness (Maity 2011) and roughness (Dubois and Prade 1987) are common factors in any real life problem including inventory control. Well established mathematical tools are available to deal with problems involving these uncertainties (Ishii and Konno 1998; Maity 2011;). But in real life, some problems occur where both fuzziness and roughness exist simultaneously. To overcome these situations normally fuzzy rough variables are used to model the problem. Dubois and Prade (1990) introduced the concept of fuzzy rough sets. After that, some researchers (Morsi and Yakout 1998; Radzikowska and Kerre 2002; etc) defined fuzzy rough set as a more general case. Liu (2002) proposed some definitions and discussed some valuable properties of fuzzy rough variable. Using this approach some researchers modelled different problems where fuzziness and roughness occur simultaneously (Xu and Zhao 2008, 2010).

Multi-item classical inventory models under resource constraints are available in well-known. In the case of a multi-item inventory model, it is easy to study each item separately as long as there are no interactions between the items. However, in general, interactions exist between such items in the form of limited warehouse capacity, limited budgetary resource, etc. Moreover in the formulation of such multi-item inventory models, the interesting terms, which changes the demand on one item due to the presence of other, till now, has been accommodated in the model by a few researchers (Maity and Maiti 2009). Now-a-days, retailers opt for multi item business as loss in one item, if it happens, may be compensated by the other. In the recent competitive market, the inventory/stock is decoratively exhibited and colourably displayed through electronic media to attract the customers and thus to push the sale. The marketing research has recognized this relationship and incorporated it into product assortment and shelf-space allocation models. These models are formulated with the demand rate as a function of the shelf-space allocated with the product and sometimes to the substitute and/or complementary products also.

From all these studies, some lacunas in the existing EOQ models on trade credit policy are found which are summarized below:

From all these studies, there are some lacunas in the development and evaluation of constrained multi-item optimal control problems with imperfect production. These gaps are:

- None have considered the inventory costs, purchasing and selling prices as fuzzy-rough parameters.
- Very few researchers have taken imperfect production process as reliability dependent and unit production cost as function of reliability parameter.
- None included the cost due to the measures taken for clean production which is must as per the several national and international regulations in the unit production cost.
- None have formulated the constrained multi-item optimal control problem with reliability under the imperfect production process using Pontryagin's optimal control.

- None have considered substitute and/ or complementary items optimal control problem with the reliability dependent imperfect production process.
- Normally, no manufacturing process is free from imperfect production. Moreover, with time, malfunctioning of machineries increases. Though the workers are experienced with longer production experience, ultimately their fatigue overcomes their expertiseness. Also, supervision over the production process is slowly loosened as the time passes. Thus, as a result, in a manufacturing system, production of defective items increases with time due to several factors as mentioned above. But this phenomenon has been overlooked by the earlier researchers.

In this investigation, an imperfect production-inventory model consisting of complementary and/or substitute type's items is considered against a resource constraint. In the case of substitute items, demands are negative due to the effect of one item on the others and it is positive when the items are of complementary types. The customers of an item may be influenced to go for a substitute item seeing the stock level of the substitute one. Again some customers may purchase the complementary items to avail the location advantage, to save time, etc. Here demands are assumed to be stock dependent. The warehouse to store the items is of limited capacity which is imprecise in nature. The relevant inventory costs like production cost, holding and set up costs and sales revenue are considered to be fuzzy rough. Here the unit production cost is extended over the Khouja (1995) consideration to include the cost due to clean production. The profit out of the total proceeds is evaluated and maximized. This maximization problem is formulated as an optimal control problem and solved following the Pontryagin's principle (Pontryagin et al. 1962) satisfying the Generalized Legendre conditions. Fuzzy rough expectation is taken over the fuzzy rough parameters and the possibility measure is considered for the fuzzy resource constraint. The model is illustrated for three items through some numerical data. Optimum production, demand and stock level are determined and presented both graphically and in tabular forms.

Preliminaries and deductions:

Fuzzy Extension principle (Zadeh 1978): If $\tilde{a}, \tilde{b} \in R$ and $\tilde{c} = f(\tilde{a}, \tilde{b})$ where $f: R \times R \to R$ be a binary operation then membership function $\mu_{\tilde{c}}$ of $\tilde{c}$ is defined as

$$\mu_{\tilde{c}}(z) = \sup\{\min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)), x, y \in R \; and \; z = f(x, y), \forall z \in R \qquad (1)$$

Possibility(Pos), Necessity(Nes) measure: Any fuzzy subset $\tilde{a}$ of R with membership function $\mu_{\tilde{a}}(x): R \to [0,1]$ is called a fuzzy number. Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy quantities with membership functions $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$ respectively. Then according to Dubois and Prade (1983, 1988), Liu and Iwamura (1998) possibility and necessity measure of a fuzzy event are defined as below:

$$Pos(\tilde{a} * \tilde{b}) = \{\sup(\min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)), x, y \in R, x * y\}, \quad (2)$$

$$Nes(\tilde{a} * \tilde{b}) = 1 - \overline{Pos(\tilde{a} * \tilde{b})} \quad (3)$$

where $*$ is any arithmetic relational operator and R is set of real numbers.

Similarly possibility and necessity measures of $\tilde{a}$ with respect to $\tilde{b}$ are denoted by $\Pi_{\tilde{b}}(\tilde{a}) = \sup\{\min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(x)), x \in R\}$ (4)

$$N_{\tilde{b}}(\tilde{a}) = \min\{\sup(\mu_{\tilde{a}}(x), 1 - \mu_{\tilde{b}}(x)), x \in R\} \quad (5)$$

Credibility measure (Liu and Liu 2003): If A be a fuzzy event then credibility measure of A is denoted by $Cr(A)$ and defined as $Cr(A) = \frac{1}{2}(Pos(A) + Nes(A))$

Fuzzy expectation (Liu and Liu 2002): Let $\tilde{A}$ be a normalized fuzzy variable.

The expected value of the fuzzy variable $\tilde{A}$ is denoted by $E(\tilde{A})$ and defined by

$$\boldsymbol{E}[\tilde{A}] = \int_0^\infty \mathrm{Cr}(\tilde{A} \geq \mathrm{r})\mathrm{dr} - \int_{-\infty}^0 \mathrm{Cr}(\tilde{A} \leq \mathrm{r})\mathrm{dr} \quad (6)$$

Provided that at least one of the two integrals is finite.

Lemma-1 (Maity 2011): If $\tilde{a} = (a_1, a_2, a_3, a_4)$ be a TrFN and b be a crisp number then $Pos(\tilde{a} \geq b) = \begin{cases} 1 & \text{if } a_3 \geq b \\ \frac{a_4 - b}{a_4 - a_3} & \text{if } a_3 \leq b \leq a_4 \\ 0 & \text{otherwise} \end{cases}$

Lemma-2 (Maity 2011): If $\tilde{a} = (a_1, a_2, a_3, a_4)$ be a TrFN and b be a crisp number then $Nes(\tilde{a} \geq b) = \begin{cases} 1 & \text{if } a_1 \geq b \\ \frac{a_2 - b}{a_2 - a_1} & \text{if } a_1 \leq b \leq a_2 \\ 0 & \text{otherwise} \end{cases}$

Lemma-3 (Maity 2011): If $\tilde{a} = (a_1, a_2, a_3, a_4)$ is a TrFN, then expected value of $\tilde{a}, E[\tilde{a}]$, is given by $E[\tilde{a}] = \frac{1}{4}[(a_1 + a_2 + a_3 + a_4)]$.

Rough space (Liu 2002): Let Λ be a non empty set, κ a σ algebra of subsets of Λ, and Δ an element in κ and π a trust measure. Then $(\Lambda, \Delta, \kappa, \pi)$ is called a rough space.

Rough variable (Liu 2002): Let $(\Lambda, \Delta, \kappa, \pi)$ be a rough space. A rough variable ξ is a measurable function from the rough space $(\Lambda, \Delta, \kappa, \pi)$ to the set of real numbers. i.e. for every Borel set B of R, $\{\lambda \in \Lambda | \xi(\lambda) \in B\} \in \kappa$. The lower $(\underline{\xi})$ and upper ($\bar{\xi}$) approximations of the rough variable ξ are given by $\underline{\xi} = \{\xi(\lambda)|\lambda \in \Delta\}$ and $\bar{\xi} = \{\xi(\lambda)|\lambda \in \Lambda\}$.

Trust measure (Liu 2002): Let $(\Lambda, \Delta, \kappa, \pi)$ be a rough space. The trust measure of event A is denoted by Tr{A} and defined by Tr{A}=$\frac{1}{2}(Tr\{A\} + Tr\{A\})$ where $\underline{\mathrm{Tr}}\{A\}$ denotes the lower trust measure of event A, defined by $\underline{\mathrm{Tr}}\{A\}$ =$\frac{\pi\{A\cap\Delta\}}{\pi\{\Delta\}}$, and $\overline{Tr}\{A\}$ denotes the upper trust measure of event A, defined by $\overline{Tr}\{A\} = \frac{\pi\{A\}}{\pi\{\Lambda\}}$,when the enough information about the measure π is not given, it may be treated as Lebesgue measure. Then we can get the trust measure of the rough event š ≥ t, Tr{ š ≥ t } and its function curve (cf. Fig-1) as presented below where t is crisp number, š is a rough variable given by š=([a,b][c,d]), 0≤c≤a≤b≤d.

$$\mathrm{Tr}\{\check{s} \geq t\} = \begin{cases} 0 & \text{for } d \leq t \\ \frac{(d-t)}{2(d-c)} & \text{for } b \leq t \leq d \\ \frac{1}{2}\left(\frac{d-t}{d-c} + \frac{b-t}{b-a}\right) & \text{for } a \leq t \leq b \\ \frac{1}{2}\left(\frac{d-t}{d-c} + 1\right) & \text{for } c \leq t \leq a \\ 1 & \text{for } t \leq c \end{cases}$$

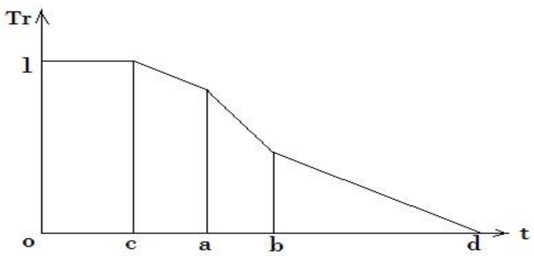

Figure-1: Tr{ š ≥ t } function curve

Rough expectation (Liu 2002): Let š be a rough variable. The expected value of the rough variable š is denoted by E[š] and defined by $\boldsymbol{E}[\check{s}] = \int_0^{\infty} \mathrm{Tr}(\check{s} \geq r)dr - \int_{-\infty}^{0} \mathrm{Tr}(\check{s} \leq r)dr$, (7)

Provided that at least one of the two integrals is finite.

Lemma-4: If š=([a,b][c,d]) is a rough variable and r is a crisp number, then expected value of š, E[š] is given by $\mathrm{E}[\check{s}] = \frac{1}{4}[a + b + c + d]$

Proof: Since š=([a,b][c,d]) is a rough variable and r is a crisp number, then from definition of trust measure we have

$$Tr\{ \check{s} \geq r \} = \begin{cases} 0 & \text{for } d \leq r \\ \frac{(d-r)}{2(d-c)} & \text{for } b \leq r \leq d \\ \frac{1}{2}\left(\frac{d-r}{d-c}+\frac{b-r}{b-a}\right) & \text{for } a \leq r \leq b \\ \frac{1}{2}\left(\frac{d-r}{d-c}+1\right) & \text{for } c \leq r \leq a \\ 1 & \text{for } r \leq c \end{cases} \qquad Tr\{ \check{s} \leq r \} = \begin{cases} 0 & \text{for } r \leq c \\ \frac{(r-c)}{2(d-c)} & \text{for } c \leq r \leq a \\ \frac{1}{2}\left(\frac{r-c}{d-c}+\frac{r-a}{b-a}\right) & \text{for } a \leq r \leq b \\ \frac{1}{2}\left(\frac{r-c}{d-c}+1\right) & \text{for } b \leq r \leq d \\ 1 & \text{for } d \leq r \end{cases}$$

So the expected value of $\check{s}$ is calculated using (7) as follows:

$$\boldsymbol{E}[\check{s}] = \int_0^{\infty} Tr(\check{s} \geq r)dr - \int_{-\infty}^{0} Tr(\check{s} \leq r)dr$$

$$= \int_0^c 1dr + \int_c^a \frac{1}{2}\left(\frac{d-r}{d-c}+1\right)dr + \int_a^b \frac{1}{2}\left(\frac{d-r}{d-c}+\frac{b-r}{b-a}\right)dr + \int_b^d \frac{d-r}{2(d-c)}dr$$

$$= \frac{1}{4}[a+b+c+d]$$

Fuzzy rough variable (Liu 2002): A fuzzy rough variable is a measurable function from a rough space (Λ, Δ, κ, π) to the set of fuzzy variables. More generally, a fuzzy rough variable is a rough variable taking fuzzy values.

Fuzzy rough expectation (Liu 2002): Let $\check{\tilde{\xi}}$ be a fuzzy rough variable. The expected value of the fuzzy rough variable $\check{\tilde{\xi}}$ is denoted by $E[\check{\tilde{\xi}}]$ and defined by

$$\boldsymbol{E}[\check{\tilde{\xi}}] = \int_0^{\infty} Tr\left(\lambda \in \Lambda | E[\tilde{X}(\lambda)] \geq r\right)dr - \int_{-\infty}^{0} Tr\left(\lambda \in \Lambda | E[\tilde{X}(\lambda)] \leq r\right)dr \tag{8}$$

Provided that at least one of the two integrals is finite.

Lemma-5: Let $\check{\tilde{\xi}} = (\check{\xi} - L_1, \check{\xi} - L_2, \check{\xi} + R_1, \check{\xi} + R_2)$ be a fuzzy rough variable, where $\check{\xi} = ([a,b][c,d])$ is a rough variable. Then expected value of $\check{\tilde{\xi}}$, $E[\check{\tilde{\xi}}]$ is given by $E\left[\check{\tilde{\xi}}\right] = \frac{1}{4}[a+b+c+d] + \frac{(R_1+R_2)-(L_1+L_2)}{4}$

Proof: Since $\check{\tilde{\xi}} = (\check{\xi} - L_1, \check{\xi} - L_2, \check{\xi} + R_1, \check{\xi} + R_2)$ be a fuzzy rough variable, where $\check{\xi} = ([a,b][c,d])$ is a rough variable then using Lemma-3 we get,

$$E\left[\check{\tilde{\xi}}\right] = E\left[\frac{1}{4}\left[\left(2\check{\xi} - L_1 - L_2\right) + \left(2\check{\xi} + R_1 + R_2\right)\right]\right] = E\left[\check{\xi} + \theta\right] \text{ where } \theta = \frac{(R_1+R_2)-(L_1+L_2)}{4}$$

Again using Lemma-4 we get, $E\left[\check{\xi} + \theta\right] = \frac{1}{4}[a+b+c+d] + \theta$ $= \frac{1}{4}[a+b+c+d] + \frac{(R_1+R_2)-(L_1+L_2)}{4}$.

Assumptions and notations for the proposed model: The following notations and assumptions are used in developing the model.

i. The number of items is *n* which are complementary / or substitute type to each other.
ii. Demand of an item is linearly stock dependent to itself and is nonlinearly influenced by the substitute (negatively) or complementary (positively) items.
iii. This is a single period inventory model with finite time horizon, *T*.

iv. $\tilde{m}$ is the maximum space available for storage.

v. Shortages are not allowed.

vi. There is no repair over the period [*0,T*].

For *i* –th (*i* = 1,2,3...,n) item, it is assumed that,

vii. $U_i(t)$ is production rate at any time *t* which depends on the reliability parameter and $0 \leq U_i(t) \leq u_i$.

viii. $x_i(t)$ is inventory level at any time *t*.

ix. a_i is the storage area for per unit item.

x. η_i is production reliability parameter which is a decision variable, which is defined as $\frac{Number\ of\ failures}{Total\ units\ of\ operating\ hours}$. The smaller value of η_i results in higher investment in technology, whereas greater value of η_i causes smaller investment for the cost of technology.

xi. η_{imax}, η_{imin} are the maximum and minimum values of η_i respectively, $0 < \eta_i < 1$.

xii. $\check{c}_{1i}(\eta_i)$ is development cost for production system and is of the form (Mettas 2000; Sana 2010;): $\check{c}_{1i}(\eta_i) = \check{\tilde{M}}_i + \check{\tilde{N}}_i e^{R_i(\eta_{imax}-\eta_i)/(\eta_i-\eta_{imin})}$ where $\check{\tilde{M}}_i$ is the fixed cost like labor, energy etc., and is independent of η_i . $\check{\tilde{N}}_i$ is the cost of technology, resource and design complexity for production when $\eta_i = \eta_{imax}$. R_i represents the difficulties in increasing reliability, which depends on the design complexity, technology and resource limitations etc.

xiii. Unit production cost, $\check{c}_{ui}$, is a function of production reliability parameter and production rate $U_i(t)$ and is of the form $\check{c}_{ui}(\eta_i, t) = \check{c}_{0i} + \frac{\check{c}_{1i}(\eta_i)}{U_i(t)} + \check{c}_{2i}U_i(t) + \check{c}_{3i}\sqrt{U_i(t)}$ where $\check{c}_{0i}$ is the fixed material cost. Second term is the development cost which is equally distributed over the production $U_i(t)$ at any time *t*. The third term is tool/die cost which is proportional to the production rate. And the fourth term is environmental cost which is proportional to the square root of production rate.

xiv. $\check{c}_{hi}$, is the holding cost per unit per unit time and $\check{c}_{si}$ is the unit selling price.

xv. Set-up cost of the system, $\check{\tilde{S}}_{ui}(\eta_i)$ depends on production reliability and is of the form (Cheng 1989): $\check{\tilde{S}}_{ui}(\eta_i) = \check{\tilde{S}}_{ui}\eta_i^{-c_i}, c_i > 0$.

xvi. The amount of defective items produced at time t is $(\delta_i - \mu_i e^{-\eta_i t})U_i(t)$, where $\delta_i - \mu_i e^{-\eta_i t} < 1$. In this production system the production of defective items increases with increase of time. Here, the fraction $\delta_i - \mu_i e^{-\eta_i t}$ increases with time t and η_i simultaneously, because almost all manufacturing system undergoes malfunctioning/unsatisfactory performance after some time. During malfunctioning, in long run process, the system shifts in-control state to out of- control state as a result the percent of defective items increase with time t. Besides it, the lower value of η_i decrease the percent of defective items.

xvii. $q_i(t)$ is the adjoint function.

Symbols $\sim$, $\smile$ and $\asymp$ that are used on the top of the above notations, indicate fuzzy parameters, rough parameters and fuzzy rough parameters respectively.

Model Formulation: A multi-item imperfect production-inventory system in fuzzy rough environment is considered with imprecise warehouse capacity. Here, for the i-th item (i=1,2,3,...,n), the rate of production is $U_i(t)$. Demand of the items is also inventory dependent. The stock level at time, t decreases due to consumption. Shortages are not allowed. The differential equations of i-th item (i=1,2,3,...,n), representing the above system during a fixed time-horizon, T is

$$\dot{X}_i(t) = (1 - \delta_i + \mu_i e^{-\eta_i t})U_i(t) - D_i\big(X_1(t), X_2(t), \dots, X_n(t)\big), X_i(0) = X_{i0} \qquad (9)$$

And $0 \le U_i(t) \le u_i$, $D_i\big(X_1(t), X_2(t), \dots, X_n(t)\big) \ge 0$, $0 \le t \le T$, where $(\,\dot{}\,)$ denotes differentiation w. r. to t.

Here, we consider an assortment problem with n items. They are either complementary items or substitute with respect to each other. Then the demand functions are,

$$D_i(X) = a_{i0} + (a_{i1}X_1 + \cdots + a_{ii-1}X_{i-1} + a_{ii} + a_{ii+1}X_{i+1} + \cdots + a_{in}X_n)X_i \qquad (10)$$

Here, $X = (X_1, X_2, \dots, X_n)$ and a_{ij} 's $(i, j = 1,2, \dots, n)$ are negative and positive for substitute and complementary items respectively.

Assuming the warehouse to be of imprecise capacity, maximization of total fuzzy rough profit consisting of sales proceeds, holding, set up and production costs leads to

$$Maximum\ \check{\tilde{J}} = \sum_{i=1}^{n} \int_0^T \left[\check{\tilde{c}}_{si} D_i\big(X(t)\big) - \check{\tilde{c}}_{hi}X_i(t) - \check{\tilde{c}}_{ui}U_i(t) - \check{\tilde{S}}_{ui}(\eta_i)\right] dt$$
$$= \int_0^T \check{\tilde{R}}(X)\, dt \qquad (11)$$

Where $\check{\tilde{R}}(X) = \sum_{i=1}^{n} \check{\tilde{R}}_i(X) = \sum_{i=1}^{n} \left[\check{\tilde{c}}_{si} D_i\big(X(t)\big) - \check{\tilde{c}}_{hi}X_i(t) - \check{\tilde{c}}_{ui}U_i(t) - \check{\tilde{S}}_{ui}(\eta_i)\right]$ (12)

Subject to (9) and $\sum_{i=1}^{n} a_i X_i(t) \le \tilde{m}$. (13)

Equivalent Deterministic Representation of the Proposed Model: The fuzzy rough selling price and different inventory costs are transformed to crisp equivalent by fuzzy rough expectation using Lemma-5. The imprecise constraint transformed to crisp equivalent following Lemma-1 by chance constraint programming approach. Then the crisp equivalent of the above problem takes to the following form:

$$Maximum\ E[\breve{\tilde{J}}] = \sum_{i=1}^{n} \int_{0}^{T} \left[E[\breve{\tilde{c}}_{si}]D_i(X(t)) - E[\breve{\tilde{c}}_{hi}]X_i(t) - E[\breve{\tilde{c}}_{ui}]U_i(t)E[\breve{\tilde{S}}_{ui}](\eta_i{}^{-c_i})\right] dt$$

$$= \int_0^T R(X)\,dt \qquad (14)$$

Where $R(X) = E[\breve{\tilde{R}}(X)]$, Subject to (9) and $\sum_{i=1}^{n} a_i X_i(t) \le M$ where $M = \theta_1 m_3 + (1-\theta_1)m_4$, since $\tilde{m} = (m_1, m_2, m_3, m_4)$ be a TrFN and θ_1 is the pre determined confidence level.

Solution Methodology: The corresponding Hamiltonian function is

$$H = R(X) + \sum_{i=1}^{n}[q_i(t)((1-\delta_i + \mu_i e^{-\eta_i t})U_i(t) - D_i(X_1(t), X_2(t), \dots, X_n(t)))] \qquad (15)$$

Where $q_i(t)$'s are the adjoint variables and the Lagrangian function is

$$L = H + \lambda(M - \sum_{i=1}^{n} a_i X_i(t))\text{, where } \lambda(\ge 0) \text{ is the Lagrange multiplier.} \qquad (16)$$

Then, the Kuhn-Tucker condition is $\lambda(M - \sum_{i=1}^{n} a_i X_i(t)) = 0$ (17)

If there exits $u^* = (u_1{}^*, u_2{}^*, \dots, u_n{}^*)$, $0 \le U_i(t) \le u_i$ for which (X^*, u^*) gives the optimal solution, then from Pontryagin's maximum principle, at (X^*, u^*) it is given that

$$\dot{q}_i(t) = -\frac{\partial L}{\partial X_i(t)} \qquad (18)$$

$$\frac{\partial H}{\partial U_i(t)} = 0 \qquad (19)$$

Solving (18), using the condition $q_i(T) = 0$ the adjoint functions $q_i(t), for\ i = 1,2, \dots, n$ are found. Putting the value of $q_i(t), for\ i = 1,2, \dots, n$ in (19) the corresponding required production function $U_i(t)$ is given.

PARTICULAR CASE

Here, an assortment problem with three items in which first is a complementary product with respect to the others and second and third is substitute products to each other, is considered. Then the demand functions are

$$D_1(X_1, X_2, X_3) = a_{10} + a_{11}X_1 + a_{12}D_2(X_1, X_2, X_3) + a_{13}D_3(X_1, X_2, X_3)$$

$$D_2(X_1, X_2, X_3) = a_{20} + a_{21}D_1(X_1, X_2, X_3) + a_{22}X_2 - a_{23}X_3$$

$$D_3(X_1, X_2, X_3) = a_{30} + a_{31}D_1(X_1, X_2, X_3) - a_{32}X_2 + a_{33}X_3$$

Solving the above demand functions, we get,

$D_1(X_1, X_2, X_3) = b_{10} + b_{11}X_1 + b_{12}X_2 + b_{13}X_3;$
$D_2(X_1, X_2, X_3) = b_{20} + b_{21}X_1 + b_{22}X_2 - b_{23}X_3;$
$D_3(X_1, X_2, X_3) = b_{30} + b_{31}X_1 - b_{32}X_2 + b_{33}X_3$; where
$b_{10} = \frac{a_{10}+a_{12}a_{20}+a_{13}a_{30}}{1-a_{12}a_{21}-a_{13}a_{31}}, b_{11} = \frac{a_{11}}{1-a_{12}a_{21}-a_{13}a_{31}}, b_{12} = \frac{a_{12}a_{22}-a_{13}a_{32}}{1-a_{12}a_{21}-a_{13}a_{31}}, b_{13} = \frac{-a_{12}+a_{23}+a_{13}a_{33}}{1-a_{12}a_{21}-a_{13}a_{31}}, b_{20} = a_{20} + b_{10}a_{21}, b_{21} = b_{11}a_{21}, b_{22} = a_{22} + b_{12}a_{21}, b_{23} = b_{13}a_{21} - a_{23}, b_{30} = a_{30} + b_{10}b_{31}, b_{31} = b_{11}a_{31}, b_{32} = b_{12}a_{31} - a_{32}, b_{33} = a_{33} + b_{13}a_{31}$
and they are all positive.

The differential equations for *i*-th (*i=1,2,3*) item representing above system during a fixed time-horizon, T is

$$\dot{X}_i(t) = (1 - \delta_i + \mu_i e^{-\eta_i t})U_i(t) - D_i\big(X_1(t), X_2(t), X_3(t)\big), X_i(0) = X_{i0} \tag{20}$$

And $0 \le U_i(t) \le u_i, D_i\big(X_1(t), X_2(t), X_3(t)\big) \ge 0, 0 \le t \le T$.

Then the corresponding Hamiltonian function is

$$H = R(X) + \sum_{i=1}^{3}[q_i(t)((1 - \delta_i + \mu_i e^{-\eta_i t})U_i(t) - D_i\big(X_1(t), X_2(t), X_3(t)\big))] \tag{21}$$

Where $q_i(t)$'s are the adjoint variables and the Lagrangian function is

$$L = H + \lambda(M - \sum_{i=1}^{3} a_i X_i(t)) \tag{22}$$

Where $\lambda(\ge 0)$ is the Lagrange multiplier.

Then, the Kuhn-Tucker condition is $\lambda(M - \sum_{i=1}^{3} a_i X_i(t)) = 0$
(23)

The corresponding adjoint functions $q_i(t)$ is given by 1st order differential equation

$\dot{q}_i(t) = -\frac{\partial L}{\partial X_i(t)}$, with $q_i(T) = 0, i = 1,2,3.$
(24)

According to the maximum principle, the Hamiltonian is maximized at every point of time with respect to admissible controllable production function $U_i(t)$. This lead to the relation,

$$\frac{\partial H}{\partial U_i(t)} = -c_{0i} - 2c_{2i}U_i(t) - \frac{3}{2}c_{3i}\sqrt{U_i(t)} + q_i(t)(1 - \partial_i + \mu_i e^{-\eta_i t}) = 0$$

Therefore, for a bounded production system, the optimum production function is given by

$$U_i^*(t) = \begin{cases} u_i & if\ q_i(t)(1 - \partial_i + \mu_i e^{-\eta_i t}) \ge c_{0i} + u_i c_{2i} - \frac{9c_{3i}^2}{16c_{2i}} \\ \frac{8c_{2i}P_i(t) + 9c_{3i}^2}{16c_{2i}^2} & if\ q_i(t)(1 - \partial_i + \mu_i e^{-\eta_i t}) \ge c_{0i} - \frac{9c_{3i}^2}{16c_{2i}} \\ 0 & else\ where \end{cases} \tag{25}$$

Where $P_i(t) = -2c_{2i} + 2q_i(t)(1 - \partial_i + \mu_i e^{-\eta_i t})$.

From equation (24), we get

$$\dot{q}_1(t) - b_{11}q_1(t) - b_{21}q_2(t) - b_{31}q_3(t) = k_{11} \quad (26)$$

$$-b_{12}q_1(t) + \dot{q}_2(t) - b_{22}q_2(t) + b_{32}q_3(t) = k_{22} \quad (27)$$

$$-b_{13}q_1(t) + b_{23}q_2(t) + \dot{q}_3(t) - b_{33}q_3(t) = k_{33} \quad (28)$$

With $q_i(T) = 0, i = 1,2,3$. And $k_{11} = c_{h1} + \lambda a_1 - c_{s1}b_{11} - c_{s2}b_{21} - c_{s3}b_{31}, k_{22} = c_{h2} + \lambda a_2 - c_{s1}b_{12} - c_{s2}b_{22} + c_{s3}b_{32}, k_{33} = c_{h3} + \lambda a_3 - c_{s1}b_{13} + c_{s2}b_{23} + c_{s3}b_{33}$

Solving the equations (26), (27) and (28) along with the boundary conditions we get $q_i(t), i = 1,2,3$.

Putting the value of $q_i(t), i = 1,2,3$, in the equation (25), the optimal production function is given by $U_i^*(t) = \begin{cases} u_i & if\ \ 0 \le t \le t_{1i} \\ \frac{8c_{2i}P_i(t) + 9c_{3i}^2}{16c_{2i}^2} & if\ \ t_{1i} \le t \le t_{2i} \\ 0 & if\ \ t_{2i} \le t \end{cases}$

Here, the optimum value of λ is obtained by trial and error method satisfying the Kuhn-Tucker condition (23) in equality sense.

Numerical Experiment:

To illustrate the above inventory model numerically, an inventory system of three items is considered with $T = 6$ units. Here $\tilde{m} = (300,310,320,330), \theta_1 = 0.8$,

$\check{\check{c}}_{h1} = (\check{c}_{h1} - 0.08, \check{c}_{h1} - 0.05, \check{c}_{h1} + 0.06, \check{c}_{h1} + 0.07)$, $\check{c}_{h1} = ([1.1,1.3][1,1.4])$;

$\check{\check{c}}_{h2} = (\check{c}_{h2} - 0.06, \check{c}_{h2} - 0.04, \check{c}_{h2} + 0.03, \check{c}_{h2} + 0.07), \check{c}_{h2} = ([1.5,1.7][1.2,2.0])$;

$\check{\check{c}}_{h3} = (\check{c}_{h3} - 0.08, \check{c}_{h3} - 0.05, \check{c}_{h3} + 0.06, \check{c}_{h3} + 0.07), \check{c}_{h3} = ([1.8,2.0][1.5,2.3])$;

$\check{\check{c}}_{s1} = (\check{c}_{s1} - 1, \check{c}_{s1} - 0.5, \check{c}_{s1} + 0.6, \check{c}_{s1} + 0.9), \check{c}_{s1} = ([10,12][9,13])$;

$\check{\check{c}}_{s2} = (\check{c}_{s2} - 1.5, \check{c}_{s2} - 1, \check{c}_{s2} + 1, \check{c}_{s2} + 1.5), \check{c}_{s2} = ([15,16][14,17])$;

$\check{\check{c}}_{s3} = (\check{c}_{s3} - 2, \check{c}_{s3} - 1.5, \check{c}_{s3} + 1, \check{c}_{s3} + 2.5), \check{c}_{s3} = ([17,19][15,21])$;

$\check{\check{c}}_{01} = (\check{c}_{01} - 0.08, \check{c}_{01} - 0.05, \check{c}_{01} + 0.06, \check{c}_{01} + 0.07), \check{c}_{01} = ([1.1,1.5][1.0,1.6])$;

$\check{\check{c}}_{02} = (\check{c}_{02} - 0.06, \check{c}_{02} - 0.04, \check{c}_{02} + 0.03, \check{c}_{02} + 0.07), \check{c}_{02} = ([1.3,1.7][1.2,1.8])$;

$\check{\check{c}}_{03} = (\check{c}_{03} - 0.08, \check{c}_{03} - 0.05, \check{c}_{03} + 0.06, \check{c}_{03} + 0.07), \check{c}_{03} = ([1.4,1.6][1.3,1.7])$;

$\breve{\tilde{c}}_{21} = (\breve{c}_{21} - 0.08, \breve{c}_{21} - 0.05, \breve{c}_{21} + 0.06, \breve{c}_{21} + 0.07), \breve{c}_{21} = ([0.02,0.04][0.01,0.05]);$

$\breve{\tilde{c}}_{22} = (\breve{c}_{22} - 0.06, \breve{c}_{22} - 0.04, \breve{c}_{22} + 0.03, \breve{c}_{22} + 0.07), \breve{c}_{22} = ([0.03,0.05][0.01,0.07]);$

$\breve{\tilde{c}}_{23} = (\breve{c}_{23} - 0.08, \breve{c}_{23} - 0.05, \breve{c}_{23} + 0.06, \breve{c}_{23} + 0.07), \breve{c}_{23} = ([0.04,0.06][0.03,0.07]);$

$\breve{\tilde{c}}_{31} = (\breve{c}_{31} - 0.08, \breve{c}_{31} - 0.05, \breve{c}_{31} + 0.06, \breve{c}_{31} + 0.07), \breve{c}_{31} = ([0.04,0.06][0.02,0.08]);$

$\breve{\tilde{c}}_{32} = (\breve{c}_{32} - 0.06, \breve{c}_{32} - 0.04, \breve{c}_{32} + 0.03, \breve{c}_{32} + 0.07), \breve{c}_{32} = ([0.07,0.09][0.05,0.11]);$

$\breve{\tilde{c}}_{33} = (\breve{c}_{33} - 0.08, \breve{c}_{33} - 0.05, \breve{c}_{33} + 0.06, \breve{c}_{33} + 0.07), \breve{c}_{33} = ([0.06,0.08][0.04,0.1]);$

$\breve{\tilde{S}}_{u1} = (\breve{S}_{u1} - 10, \breve{S}_{u1} - 5, \breve{S}_{u1} + 6, \breve{S}_{u1} + 9), \breve{S}_{u1} = ([100,140][90,150]);$

$\breve{\tilde{S}}_{u2} = (\breve{S}_{u2} - 15, \breve{S}_{u2} - 10, \breve{S}_{u2} + 10, \breve{S}_{u2} + 15), \breve{S}_{u2} = ([115,145][100,160]);$

$\breve{\tilde{S}}_{u3} = (\breve{S}_{u3} - 20, \breve{S}_{u3} - 15, \breve{S}_{u3} + 10, \breve{S}_{u3} + 25), \breve{S}_{u3} = ([135,165][115,185]);$

$\breve{\tilde{M}}_1 = (\breve{M}_1 - 1, \breve{M}_1 - 0.5, \breve{M}_1 + 0.6, \breve{M}_1 + 0.9), \breve{M}_1 = ([14,16][12,18]);$

$\breve{\tilde{M}}_2 = (\breve{M}_2 - 1.5, \breve{M}_2 - 1, \breve{M}_2 + 1, \breve{M}_2 + 1.5), \breve{M}_2 = ([15,16][11,18]);$

$\breve{\tilde{M}}_3 = (\breve{M}_3 - 2, \breve{M}_3 - 1.5, \breve{M}_3 + 1, \breve{M}_3 + 2.5), \breve{M}_3 = ([15,16][12,17]);$

$\breve{\tilde{N}}_1 = (\breve{N}_1 - 1, \breve{N}_1 - 0.5, \breve{N}_1 + 0.6, \breve{N}_1 + 0.9), \breve{N}_1 = ([11,13][10,14]);$

$\breve{\tilde{N}}_2 = (\breve{N}_2 - 1.5, \breve{N}_2 - 1, \breve{N}_2 + 1, \breve{N}_2 + 1.5), \breve{N}_2 = ([12,13][8,15]);$

$\breve{\tilde{N}}_3 = (\breve{N}_3 - 2, \breve{N}_3 - 1.5, \breve{N}_3 + 1, \breve{N}_3 + 2.5), \breve{N}_3 = ([11,12][10,15]);$

The other relevant input data are presented in Table-1.

Table-1. Input data

$item\,(i)$	a_{i0}	a_{i1}	a_{i2}	a_{i3}	δ_i	μ_i	R_i	u_i	a_i
1	4	0.2	0.05	0.04	0.99	0.89	0.5	100	1.5
2	5	0.04	0.3	0.06	0.99	0.84	0.5	100	1.7
3	4	0.02	0.04	0.4	0.99	0.83	0.5	50	1.7

For these input data, we find the optimal value of λ is 10 and $X_i(t), U_i(t)\, and\ D_i(X)$ are evaluated for different values of t. The values of these functions are presented in Table-2 and are depicted in Figures-2, -3 and -4. The optimum expected profit and the corresponding reliability indicators are obtained and also presented in Table-3.

Table 2: Optimal values of $X_i(t), U_i(t)\, and\ D_i(X)$

t	$X_1(t)$	$X_2(t)$	$X_3(t)$	$U_1(t)$	$U_2(t)$	$U_3(t)$	$D_1(X(t)$	$D_2(X(t)$	$D_3(X(t)$
0	180	10	20	100	100	50	136.84	81.86	30.38
2.5	101.47	33.01	27.12	100	100	50	107.65	64.90	37.79
4.63	42.98	54.71	41.90	100	100	50	95.11	54.11	47.11
5	32.88	57.97	39.59	100	100	40.68	87.74	52.76	40.68
5.14	29.65	59.01	36.84	100	100	25.63	83.57	52.63	36.93
5.2	28.37	59.18	35.43	100	86.82	19.91	81.53	52.46	35.22
5.36	25.39	57.39	31.20	100	53.84	6.16	75.11	50.75	30.82
5.4	24.76	56.64	30.22	89.66	47.40	3.51	73.52	50.14	29.88
5.45	23.60	55.42	28.84	75.93	38.84	0	71.08	49.08	28.49
5.55	20.36	52.35	26.11	49.31	22.22	0	65.35	46.15	25.48
5.7	13.84	46.68	22.58	13.85	0	0	55.81	40.32	20.99
5.73	12.46	45.56	22.00	7.85	0	0	53.98	39.12	21.17
5.77	10.54	44.11	21.26	0	0	0	51.51	37.51	19.06
5.85	6.53	41.17	19.79	0	0	0	46.48	34.19	16.79
6	0.17	36.45	17.55	0	0	0	38.56	28.87	13.27

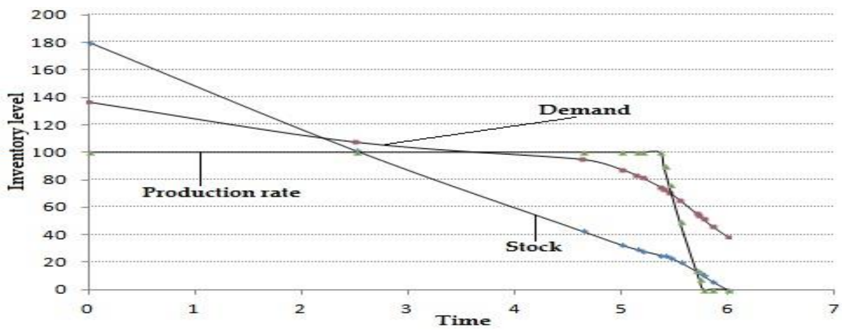

Figure-2: Optimal production, stock level and demand for complementary item-1

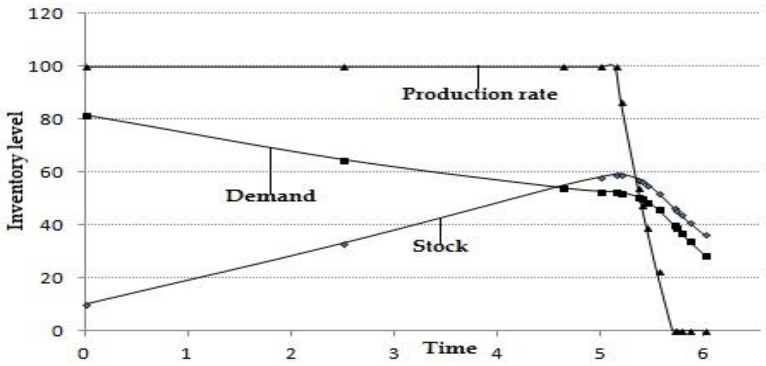

Figure-3: Optimal production, stock level and demand for substitute item-1

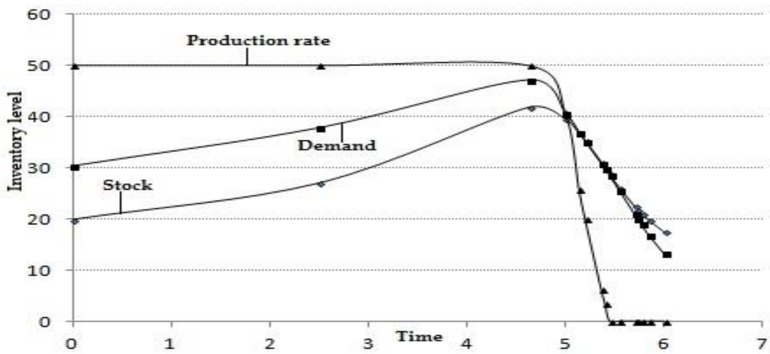

Figure-4: Optimal production, stock level and demand for substitute item-2

Table-3: Optimal values of η_i and expected profit

η_1	η_2	η_3	Expected Profit
0.2607633	0.1926742	0.1897810	3473.87

Discussion: From Figures-2, -3 and -4 it is observed that for items being complementary / substitute type, demands of these items increasing / decreasing to due to one another's influence.

Conclusion: The model has proposed optimum production inventory policies for a multi-item production inventory system with space constraint in uncertain environment. For the first time fuzzy-rough expectation is introduced and applied. Here, a multi-item system of complementary and substitute types with fuzzy rough inventory costs and selling price under a fuzzy resource constraint has been formulated and solved via optimal control theory. The formulation and analysis presented here are quite general and can be extended to other production-inventory problems with different types of demand, deterioration, etc. Though the models have been illustrated for three items only, it can be extended to include any number of complementary and / or substitute items.

The supply chain models with fuzzy rough costs/ parameters can be solved with the presented methodology.

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