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A Study of Finite Element Method for Laplace Equation

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Abstract. This paper aims at treating the Finite element method (FEM) for the discretization of elliptic Partial differential equation (PDE) and focuses on FEM for solving two-dimensional Laplace equation in a sub-set of a square domain.The idea is to use finite element spaces that are induced by triangulations of a square domain to discretize the two dimensional elliptic Laplace equation on the surface. Then the two numerical solutions obtained by FEM based on the number of finite elements are compared to check the accuracy of the developed scheme. The FEM MATLAB Programming is used for the solution of two dimensional Laplace equations. Results are then compared with the analytic solution to check the accuracy of the developed scheme.

Keywords: Finite Element Model, MATLAB Programming, Dirichlet Boundary Conditions, PDE, Finite Difference Method (FDM, Laplace Equation.

I. INTRODUCTION

It is conventional to solve Laplace Equation [1] in two dimensions with Dirichlet conditions. The finite element method (FEM), sometimes referred to as finite element analysis (FEA), is a computational technique used to obtain approximate solutions of boundary value problems in engineering. Simply stated, a boundary value problem is a mathematical problem in which one or more dependent variables must satisfy a differential equation every where within a known domain of independent variables and satisfy specific conditions on the boundary of the domain. Boundary value problems are also sometimes called field problems.The Dirichlet and Neumann boundary value problems of Laplace equation are included in advanced courses [2].Finite element methods use simple piecewise functions (e.g. linear or quadratic) valid on elements to describe the local variations of unknown flow variables ϕ. Two dimensional Laplace equation with Dirichlet boundary conditions is a model equation for steady state distribution of heat in a plane region [3]. Elliptic equations are governed by conditions on the boundary of closed domain. We consider here one of the most commonly encountered elliptic equations, namely, Laplace equation in the following form: $u_{xx} + u_{yy} = 0$.To solve the equation using Finite Element method, boundary conditions are require, such as Dirichlet's boundary conditions or Cauchy's boundary conditions. For solving the Laplace equation, we consider the following Dirichlet's boundary conditions
$u = 0$, for $0 \leq x \leq 2, y = 0$ and $0 \leq y \leq 2, x = 0$,
$u = x$, for $0 \leq x \leq 1, y = 2$,
$u = y$, for $0 \leq y \leq 1, x = 2$,
$u = 1$, along the diagonal boundary.
The above considerable region is the sub-region of the square $0 \leq x \leq 2$ and $0 \leq y \leq 2$.
We will solve the Laplace's equation for this sub-region with the help of Finite Element Method. As a result we obtain a set of algebraic equations for the unknown coefficients of the approximating functions [4, 7].

1.2. Applications of the Finite Element Method

The finite element method can be used to analyze both structural and nonstructural problems. Typical structural areas include

1. Stress analysis, including truss and frame analysis, and stress concentration problems typically associated with holes, fillets, or other changes in geometry in a body, 2. Buckling, 3. Vibration analysis

Nonstructural problems include 4. Heat transfer, 5. Fluid flow, including seepage through porous media, 6. Distribution of electric or magnetic potential

1.3 Defining the Sub regions

The choice of the geometry of the sub regions is influenced by the fact that a basis function will be defined corresponding to each node; the minimization of the functional $I[U]$then involves the integrals of products of partials derivatives of these basis functions. Triangular sub regions are convenient for several regions. Triangles allow great flexibility in covering an irregularly shaped region. The use of triangular sub regions also simplifies the computation of the basis functions and the coefficients in the linear combinations of basis functions for the finite element solution of the PDE.In order to specify the sub division of the region, we must know that the location of the nodes (which are the vertices of the triangular sub regions) and also the definition of each triangular sub region in terms of the vertices that defines the triangle. We also need to distinguish between the nodes on the boundary of the region and interior nodes. As indicated above, we assume that there are m nodes, denoted V_j, with nodes$j = 1, \dots, n$ in the interior of R is divided into p triangular sub-regions $T_1, T_2, \dots, T_p$.

1.4. Defining the Basis Functions

We now assume that the basis functions ϕ_j.There is a basis function corresponding to each node. We define ϕ_j has the following property:

$\phi_j = 1$ at node j

$\phi_j = 0$ at node $k, j \neq k$.

ϕ_j is linear on each triangular sub-domain.

Thus, ϕ_j is piecewise planar. For each triangle that has node j as a vertex, we find a plane $z = a + bx + cy$ such that $z = 1$ at node j and $z = 0$ at the other two vertices; it is zero on any sub-domain that does not have node j as a vertex.

II. SOLUTION OF LAPLACE EQUATION USING FEM:

The region $0 \leq x \leq 2$ and $0 \leq y \leq 2$, represents the square region and we consider a subset R of the square$0 \leq x \leq 2$ and $0 \leq y \leq 2$. The sub-region$R$ is decomposed into 14 triangular sub-regions, designated $T_1, T_2, T_3, \dots \dots \dots, T_{13}, T_{14}$.The vertices of these regions are the nodes $\boldsymbol{V_1}, \boldsymbol{V_2}, \dots \dots \dots, \boldsymbol{V_{11}}, \boldsymbol{V_{12}}$,as shown in Figure 2.1.

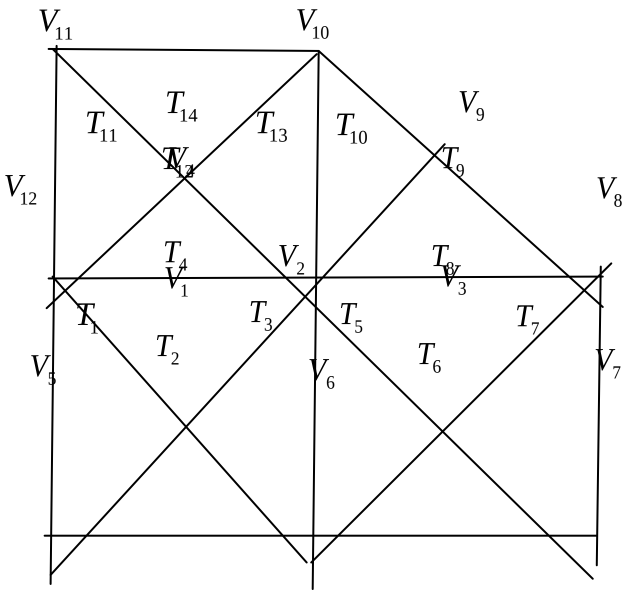


Figure2.1.A subset of square, $0 \leq x \leq 2, 0 \leq y \leq 2$.

There are four interior nodes V_1, V_2, V_3, V_4.The coordinates of the nodes are given in the matrix $\boldsymbol{V}$. The triangles$\boldsymbol{T}$, are defined by specifying the nodes indices of the three vertices of each triangle. Thus,

$$\boldsymbol{V} = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 1 \\ 3/2 & 1/2 \\ 1/2 & 3/2 \\ 0 & 0 \\ 1 & 0 \\ 2 & 0 \\ 2 & 1 \\ 3/2 & 3/2 \\ 1 & 2 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{T} = \begin{bmatrix} 1 & 5 & 12 \\ 1 & 5 & 6 \\ 1 & 2 & 6 \\ 1 & 2 & 12 \\ 3 & 2 & 6 \\ 3 & 6 & 7 \\ 3 & 7 & 8 \\ 3 & 8 & 2 \\ 2 & 8 & 9 \\ 2 & 9 & 10 \\ 4 & 11 & 12 \\ 4 & 12 & 2 \\ 4 & 2 & 10 \\ 4 & 10 & 11 \end{bmatrix}$$

2.1. Finding the Basis Functions:

We now find the twelve basis functions for the region Rdescribed in the figure 2.1.
There are twelve nodes and each node has a unique basis function
ϕ_j for $j = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$; such that
$\phi_j = 1$ at node j
$\phi_j = 0$ at node $k, j \neq k$.
ϕ_j is linear on each triangular sub-domain.
Thus, ϕ_j is piecewise planar. For each triangle that has node j as a vertex, we find a plane $z = a + bx + cy$ such that $z = 1$ at node j and $z = 0$ at the other two vertices; it is zero on any sub-domain that does not have node j as a vertex.

<u>The 1st Basis function:</u>

$\phi_j = 1$ at node 1 and $\phi_1 = 0$ at nodes 2,3,4,5,6,7,8,9,10,11,and 12. On T_1, ϕ_1 is determined by nodes 1,5,and 12; $i.e.$, at $(1/2,12), (0,0),$ and $(0,1)$.

$1 = a + \frac{1}{2}b + \frac{1}{2}c, 0 = a + 0b + 0c,$and $0 = a + 0b + c$
$\Rightarrow a = 0, b = 2,$and $c = 0$.

On T_2, ϕ_1 is determined by nodes 1,5,and 7; $i.e.$, at $(1/2,1/2), (0,0)$ and $(1,0)$.

$1 = a + \frac{1}{2}b + \frac{1}{2}c, 0 = a + 0b + 0c,$ and $0 = a + b + 0c$
$\Rightarrow a = 0, b = 0,$and $c = 2$

On T_3, ϕ_1 is determined by nodes 1,2,and 6; $i.e.$, at $(1/2,1/2), (1,1)$and$(1,0)$.

$$1 = a + \frac{1}{2}b + \frac{1}{2}c, 0 = a + b + c, \text{and } 0 = a + b + 0c$$

$\Rightarrow a = 2, b = -2,$and $c = 0$.

On T_4, ϕ_1 is determined by nodes 1,2,and 12; $i.e.$, at
$(1/2,1/2), (1,1),$ and $(0,1)$.

$1 = a + \frac{1}{2}b + \frac{1}{2}c, 0 = a + b + c,$and $0 = a + 0b + c$
$\Rightarrow a = 2, b = 0,andc = -2$.

Since node 1 is not corner of triangles $T_5, T_6, T_7, T_8, T_9, T_{10}, T_{11}, T_{12}, T_{13}, T_{14}$. Thus

Table 2.1.Basis Function for first Node

1st basis function	T_1	T_2	T_3	T_4
ϕ_1	$2x$	$2y$	$2-2x$	$2-2y$
T_5	T_6	T_7	T_8	T_9
0	0	0	0	0
T_{10}	T_{11}	T_{12}	T_{13}	T_{14}
0	0	0	0	0

<u>The 2nd Basis function:</u>

$\phi_2 = 1$ at node 2

and $\phi_2 = 0$ at nodes 1,3,4,5,6,7,8,9,10,11, and 12.On T_3, $\phi_2$is determined by nodes 1,2,and 6; $i.e.$, at (1/2,1/2), (1,1),and(1,0).

$0 = a + \frac{1}{2}b + \frac{1}{2}c, 1 = a + b + c,$and $0 = a + b + 0c$

$\Rightarrow a = -1, b = 1,$and $c = 1.$

On T_4, $\phi_2$is determined by nodes 1,2,and 12; $i.e.$,

at(1/2,1/2), (1,1), and(0,1).

$0 = a + \frac{1}{2}b + \frac{1}{2}c, 1 = a + b + c,$and $0 = a + 0b + c$

$\Rightarrow a = -1, b = 1,$and $c = 1.$

OnT_5, ϕ_2 is determined by nodes 2,3,and 6; $i.e.$,

at (1,1), (3/2,1/2),and(1,0).

$1 = a + b + c, 0 = a + \frac{3}{2}b + \frac{1}{2}c,$and $0 = a + b + 0c$

$\Rightarrow a = 1, b = -1,$and $c = 1.$

OnT_8, ϕ_2 is determined by nodes 2,3,and 8; $i.e.$,

at (1,1), (3/2,1/2)and (2,1).

$1 = a + b + c, 0 = a + \frac{3}{2}b + \frac{1}{2}c,$and $0 = a + 2b + c$

$\Rightarrow a = 1, b = -1,$and $c = 1.$

OnT_9, ϕ_2 is determined by nodes 2,8,and 9; $i.e.$,

at (1,1), (2,1)and (3/2,3/1).

$1 = a + b + c, 0 = a + 2b + c,$and $0 = a + \frac{3}{2}b + \frac{3}{2}c$

$\Rightarrow a = 3, b = -1,$and $c = -1.$

OnT_{10}, $\phi_2$is determined by nodes 2,9,and 10; $i.e.$, at (1,1), (3/2,3/2),and(1,2).

$1 = a + b + c, 0 = a + \frac{3}{2}b + \frac{3}{2}c,$and $0 = a + b + 2c$

$\Rightarrow a = 3, b = -1,$and $c = -1.$

OnT_{12}, $\phi_2$is determined by nodes 2,4,and 12; $i.e.$, at (1,1), (1/2,3/2),and(0,1).

$1 = a + b + c, 0 = a + \frac{1}{2}b + \frac{3}{2}c,$and $0 = a + 0b + c$

$\Rightarrow a = 1, b = 1,$and $c = -1.$

On T_{13}, ϕ_2 is determined by nodes 2,4,and 10; $i.e.$, at (1,1), (1/2,3/2),and(1,2).

$1 = a + b + c, 0 = a + \frac{1}{2}b + \frac{3}{2}c,$and $0 = a + b + 2c$

$\Rightarrow a = 1, b = 1,$and $c = -1.$

Since node 2 is not corner of triangles $T_1, T_2, T_6, T_7, T_{11}, T_{14}$.Thus

Table 2.2.Basis Function for first Node

2nd basis function	T_1	T_2	T_3	T_4
ϕ_2	0	0	$-1+x+y$	$-1+x+y$
T_5	T_6	T_7	T_8	T_9
$1-x+y$	0	0	$1-x+y$	$3-x-y$
T_{10}	T_{11}	T_{12}	T_{13}	T_{14}
$3-x-y$	0	$1+x-y$	$1+x-y$	0

The 3rd Basis function:

$\phi_3 = 1$ at node 3

and $\phi_3 = 0$ at nodes 1,2,4,5,6,7,8,9,10,11, and 12. On T_5, $\phi_3$is determined by nodes 2,3,and 6; $i.e.$, at (1,1), (3/2,1/2),and(1,0).

$0 = a + b + c, 1 = a + \frac{3}{2}b + \frac{1}{2}c,$and $0 = a + b + 0c$

$\Rightarrow a = -2, b = 2,$and $c = 0.$

On T_6, $\phi_3$is determined by nodes 3,6,and 7; $i.e.$

at $\left(\frac{\frac{3}{2,1}}{2}\right)$, (1,0),and(2,0).

$1 = a + \frac{3}{2}b + \frac{1}{2}c, 0 = a + b + 0c,$

and $0 = a + 2b + 0c$

$\Rightarrow a = 0, b = 0,$and $c = 2.$

On T_7, $\phi_3$is determined by nodes 3,7,and 8; $i.e.$,

at (3/2,1/2), (2,0),and(2,1).

$1 = a + \frac{3}{2}b + \frac{1}{2}c, 0 = a + 2b + 0c,$and $0 = a + 2b + c$

$\Rightarrow a = 4, b = -2,$and $c = 0.$

On T_8, $\phi_3$is determined by nodes 2,3,and 8; $i.e.$, at (1,1), (3/2,1/2),and(2,1).

$$1 = a + b + c, 1 = a + \frac{3}{2}b + \frac{1}{2}c,\text{and } 0 = a + 2b + c$$

$$\Rightarrow a = 2, b = 0,\text{and } c = -2.$$

Since node 3 is not corner of triangles $T_1, T_2, T_3, T_4, T_9, T_{10}, T_{11}, T_{12}, T_{13}, T_{14}$.Thus

Table 2.3.Basis Function for first Node

3rd basis function	T_1	T_2	T_3	T_4
ϕ_3	0	0	0	0
T_5	T_6	T_7	T_8	T_9
$-2+2x$	$2y$	$4-2x$	$2-2y$	0
T_{10}	T_{11}	T_{12}	T_{13}	T_{14}
0	0	0	0	0

The 4th Basis function:

$\phi_4 = 1$ at node 4,

and $\phi_4 = 0$ at nodes 1,2,3,5,6,7,8,9,10,11, and 12.

On T_{11}, $\phi_4$is determined by nodes 4,11,and 12; $i.e.$,

at (1/2,3/2), (0,2),and(0,1).

$1 = a + \frac{1}{2}b + \frac{3}{2}c, 0 = a + 0b + 2c,$and $0 = a + 0b + c$

$\Rightarrow a = 0, b = 2,$and $c = 0.$

On T_{12}, $\phi_4$is determined by nodes 2,4,and 12; $i.e.$,

at (1,1), (1/2,3/2),and(0,1).

$0 = a + b + c, 1 = a + \frac{1}{2}b + \frac{3}{2}c,$and $0 = a + 0b + c$

$\Rightarrow a = -2, b = 0,$and $c = 2.$

On T_{13}, $\phi_4$is determined by nodes 2,4,and 10; $i.e.$,

at (1,1), (1/2,3/2),and(1,2).

$0 = a + b + c, 1 = a + \frac{1}{2}b + \frac{3}{2}c,$and $0 = a + b + 2c$

$\Rightarrow a = 2, b = -2,$and $c = 0.$

On T_{14}, $\phi_4$is determined by nodes 4,10,and 11; $i.e.$,

at (1/2,3/2), (1,2),and(0,2).

$1 = a + \frac{1}{2}b + \frac{3}{2}c, 0 = a + b + 2c,$and $0 = a + 0b + 2c$

$\Rightarrow a = 4, b = 0,$and $c = -2.$

Since node 4 is not corner of triangles $T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9, T_{10}$. Thus

Table 2.4.Basis Function for first Node

4th basis function	T_1	T_2	T_3	T_4
ϕ_4	0	0	0	0
T_5	T_6	T_7	T_8	T_9
0	0	0	0	0
T_{10}	T_{11}	T_{12}	T_{13}	T_{14}
0	$2x$	$-2+2y$	$2-2x$	$4-2y$

<u>The 5th Basis function:</u>

$\phi_5 = 1$ at node 5,

and $\phi_5 = 0$ at nodes 1,2,3,4,6,7,8,9,10,11, and 12.

On T_1, ϕ_5 is determined by nodes 1,5,and 12; *i.e.*,

at (1/2,1/2), (0,0),and(0,1).

$0 = a + \frac{1}{2}b + \frac{1}{2}c, 1 = a + 0b + 0c,$and $0 = a + 0b + c$

$\Rightarrow a = 1, b = -1,$and $c = -1.$

On T_2, ϕ_5 is determined by nodes 1,5,and 6; *i.e.*,

at (1/2,1/2), (0,0),and(1,0).

$0 = a + \frac{1}{2}b + \frac{1}{2}c, 1 = a + 0b + 0c,$and $0 = a + b + 0c$

$\Rightarrow a = 1, b = -1,$and $c = -1.$

Since node 5 is not corner of triangles $T_3, T_4, T_5, T_6, T_7, T_8, T_9, T_{10}, T_{11}, T_{12}, T_{13}, T_{14}$.Thus

Table 2.5.Basis Function for first Node

5th basis function	T_1	T_2	T_3	T_4
ϕ_5	$1-x-y$	$1-x-y$	0	0
T_5	T_6	T_7	T_8	T_9
0	0	0	0	0
T_{10}	T_{11}	T_{12}	T_{13}	T_{14}
0	0	0	0	0

<u>The 6th Basis function:</u>

$\phi_6 = 1$ at node 6,

and $\phi_6 = 0$ at nodes 1,2,3,4,5,7,8,9,10,11, and 12.

On T_2, ϕ_6 is determined by nodes 1,5,and 6; *i.e.*,

at (1/2,1/2), (0,0),and(1,0).

$0 = a + \frac{1}{2}b + \frac{1}{2}c, 0 = a + 0b + 0c,$and $1 = a + b + 0c$

$\Rightarrow a = 0, b = 1,$and $c = -1.$

On T_3, ϕ_6 is determined by nodes 1,2,and 6; *i.e.*

, at (1/2,1/2), (1,1),and(1,0).

$0 = a + \frac{1}{2}b + \frac{1}{2}c, 0 = a + b + c,$and $1 = a + b + 0c$

$\Rightarrow a = 0, b = 1,$and $c = -1.$

On T_5, ϕ_6 is determined by nodes 2,3,and 6; *i.e.*,

at (1,1), (3/2,1/2),and(1,0).

$0 = a + b + c, 0 = a + \frac{3}{2}b + \frac{1}{2}c,$and $1 = a + b + 0c$

$\Rightarrow a = 2, b = -1,$and $c = -1.$

On T_6, ϕ_6 is determined by nodes 3,6,and 7; *i.e.*,

at (3/2,1/2), (1,0),and(2,0).

$0 = a + \frac{3}{2}b + \frac{1}{2}c, 1 = a + b + 0c,$and $0 = a + 2b + 0c$

$\Rightarrow a = 2, b = -1,$and $c = -1.$

Since node 6 is not corner of triangles $T_1, T_4, T_7, T_8, T_9, T_{10}, T_{11}, T_{12}, T_{13}, T_{14}$.Thus

Table 2.6.Basis Function for first Node

6th basis function	T_1	T_2	T_3	T_4
ϕ_6	0	$x-y$	$x-y$	0
T_5	T_6	T_7	T_8	T_9
$2-x-y$	$2-x-y$	0	0	0
T_{10}	T_{11}	T_{12}	T_{13}	T_{14}
0	0	0	0	0

The 7th Basis function:

$\phi_7 = 1$ at node 7,

and $\phi_7 = 0$ at nodes 1,2,3,4,5,6,8,9,10,11, and 12.

On T_6, $\phi_7$is determined by nodes 3,6,and 7; $i.e.$,

at (3/2,1/2), (1,0),and(2,0).

$0 = a + \frac{3}{2}b + \frac{1}{2}c, 0 = a + b + 0c$,and $1 = a + 2b + 0c$

$\Rightarrow a = -1, b = 1$,and $c = -1$.

On T_7, $\phi_7$is determined by nodes 3,7,and 8; $i.e.$,

at (3/2,1/2), (2,0),and(2,1).

$$0 = a + \frac{3}{2}b + \frac{1}{2}c, 1 = a + 2b + 0c, \text{and } 0 = a + 2b + c$$

$$\Rightarrow a = -1, b = 1, \text{and } c = -1.$$

Since node 7 is not corner of triangles $T_1, T_2, T_3, T_4, T_5, T_8, T_9, T_{10}, T_{11}, T_{12}, T_{13}, T_{14}$.Thus

Table 2.7.Basis Function for first Node

7th basis function	T_1	T_2	T_3	T_4
ϕ_7	0	0	0	0
T_5	T_6	T_7	T_8	T_9
0	$-1+x-y$	$-1+x-y$	0	0
T_{10}	T_{11}	T_{12}	T_{13}	T_{14}
0	0	0	0	0

The 8th Basis function:

$\phi_8 = 1$ at node 8

and $\phi_8 = 0$ at nodes 1,2,3,4,5,6,7,9,10,11, and 12.

On T_7, $\phi_8$is determined by nodes 3,7,and 8; $i.e.$, at (3/2,1/2), (2,0),and(2,1).

$0 = a + \frac{3}{2}b + \frac{1}{2}c, 0 = a + 2b + 0c$,and $1 = a + 2b + c$

$\Rightarrow a = -2, b = 1$,and $c = 1$.

On T_8, $\phi_8$is determined by nodes 2,3,and 8; $i.e.$, at (1,1), (3/2,1/2),and(2,1).

$0 = a + b + c, 0 = a + \frac{3}{2}b + \frac{1}{2}c$,and $1 = a + 2b + c$

$\Rightarrow a = -2, b = 1$,and $c = 1$.

On T_9, $\phi_8$is determined by nodes 2,8,and 9; $i.e.$, at (1,1), (2,1),and(3/2,3/2).

$0 = a + b + c, 1 = a + 2b + c$,and $0 = a + \frac{3}{2}b + \frac{3}{2}c$

$\Rightarrow a = 0, b = 1$,and $c = -1$.

Since node 8 is not corner of triangles $T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_9, T_{10}, T_{11}, T_{12}, T_{13}, T_{14}$.Thus

Table 2.8. Basis Function for first Node

8thbasis function	T_1	T_2	T_3	T_4
ϕ_8	0	0	0	0
T_5	T_6	T_7	T_8	T_9
0	0	$-2+x+y$	$-2+x+y$	$x-y$
T_{10}	T_{11}	T_{12}	T_{13}	T_{14}

0	0	0	0	0

The 9th Basis function:

$\phi_9 = 1$ at node 9

and $\phi_9 = 0$ at nodes 1,2,3,4,5,6,7,8,10,11, and 12.

On T_9, ϕ_9 is determined by nodes 2,8,and 9; *i.e.*, at (1,1), (2,1),and(3/2,3/2).

$0 = a + b + c, 0 = a + 2b + c,$and $1 = a + \frac{3}{2}b + \frac{3}{2}c$

$\Rightarrow a = -2, b = 0,$and $c = 2.$

On T_{10}, ϕ_9 is determined by nodes 2,9,and 10; *i.e.*, at (1,1), (3/2,3/2),and(1,2).

$0 = a + b + c, 1 = a + \frac{3}{2}b + \frac{3}{2}c,$and $0 = a + b + 2c$

$\Rightarrow a = -2, b = 2,$and $c = 0.$

Since node 9 is not corner of triangles $T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_{11}, T_{12}, T_{13}, T_{14}$. Thus

Table 2.9.Basis Function for first Node

9th basis function	T_1	T_2	T_3	T_4
ϕ_9	0	0	0	0
T_5	T_6	T_7	T_8	T_9
0	0	0	0	$-2+2y$
T_{10}	T_{11}	T_{12}	T_{13}	T_{14}
$-2+2x$	0	0	0	0

The 10th Basis function:

$\phi_{10} = 1$ at node 10

and $\phi_{10} = 0$ at nodes 1,2,3,4,5,6,7,8,9,11, and 12.

On T_{10}, ϕ_{10} is determined by nodes 2,9,and 10; *i.e.*, at (1,1), (3/2,3/2),and(1,2).

$0 = a + b + c, 0 = a + \frac{3}{2}b + \frac{3}{2}c,$and $1 = a + b + 2c$

$\Rightarrow a = 0, b = -1,$and $c = 1.$

On T_{13}, ϕ_{10} is determined by nodes 2,4,and 10; *i.e.*, at (1,1), (1/2,3/2),and(1,2).

$0 = a + b + c, 0 = a + \frac{1}{2}b + \frac{3}{2}c,$and $1 = a + b + 2c$

$\Rightarrow a = -2, b = 1,$and $c = 1.$

On T_{14}, ϕ_{10} is determined by nodes 4,10,and 11; *i.e.*, at (1/2,3/2), (1,2),and(0,2).

$0 = a + \frac{1}{2}b + \frac{3}{2}c, 1 = a + b + 2c,$and $0 = a + 0b + 2c$

$\Rightarrow a = -2, b = 1,$and $c = 1.$

Since node 10 is not corner of triangles $T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9, T_{11}, T_{12}$.Thus

Table 2.10.Basis Function for first Node

10th basis function	T_1	T_2	T_3	T_4
ϕ_{10}	0	0	0	0
T_5	T_6	T_7	T_8	T_9
0	0	0	0	0
T_{10}	T_{11}	T_{12}	T_{13}	T_{14}
$-x+y$	0	0	$-2+x+y$	$-2+x+y$

The 11th Basis function:

$\phi_{11} = 1$ at node 11

and $\phi_{11} = 0$ at nodes 1,2,3,4,5,6,7,8,9,10, and 12.

On T_{11}, ϕ_{11} is determined by nodes 4,11,and 12; *i.e.*, at (1/2,3/2), (0,2),and(0,1).

$0 = a + \frac{1}{2}b + \frac{3}{2}c, 1 = a + 0b + 2c,$and $0 = a + 0b + c$

$\Rightarrow a = -1, b = -1,$and $c = 1.$

On T_{14}, ϕ_{11} is determined by nodes 4,10,and 11; *i.e.*, at (1/2,3/2), (1,2),and(0,2).

$0 = a + \frac{1}{2}b + \frac{3}{2}c, 0 = a + b + 2c,$and $1 = a + 0b + 2c$
$\Rightarrow a = -1, b = -1,$and $c = 1.$
Since node 11 is not corner of triangles $T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9, T_{10}, T_{12}, T_{13}$.Thus

Table 2.11.Basis Function for eleventh Node.

11^{th}basis function	T_1	T_2	T_3	T_4
ϕ_{11}	0	0	0	0
T_5	T_6	T_7	T_8	T_9
0	0	0	0	0
T_{10}	T_{11}	T_{12}	T_{13}	T_{14}
0	$-1-x+y$	0	0	$-1-x+y$

<u>The 12th Basis function:</u>
$\phi_{12} = 1$ at node 12
and $\phi_{12} = 0$ at nodes 1,2,3,4,5,6,7,8,9,10, and 11.
On T_1, $\phi_{12}$is determined by nodes 1,5,and 12; $i.e.$, at (1/2,1/2), (0,0),and(0,1).
$0 = a + \frac{1}{2}b + \frac{1}{2}c, 0 = a + 0b + 0c,$and $1 = a + 0b + c$
$\Rightarrow a = 0, b = -1,$and $c = 1.$
On T_4, $\phi_{12}$is determined by nodes 1,2,and 12; $i.e.$, at (1/2,1/2), (1,1),and(0,1).
$0 = a + \frac{1}{2}b + \frac{1}{2}c, 0 = a + b + c,$and $1 = a + 0b + c$
$\Rightarrow a = 0, b = -1,$and $c = 1.$
On T_{11}, $\phi_{12}$is determined by nodes 4,11,and 12; $i.e.$, at (1/2,3/2), (0,2),and(0,1).
$0 = a + \frac{1}{2}b + \frac{3}{2}c, 0 = a + 0b + 2c,$and $1 = a + 0b + c$
$\Rightarrow a = 2, b = -1,$and $c = -1.$
On T_{12}, $\phi_{12}$is determined by nodes 2,4,and 12; $i.e.$, at (1,1), (1/2,3/2),and(0,1).
$0 = a + b + c, 0 = a + \frac{1}{2}b + \frac{3}{2}c,$and $1 = a + 0b + c$
$\Rightarrow a = 2, b = -1,$and $c = -1.$
Since node 12 is not corner of triangles $T_2, T_3, T_5, T_6, T_7, T_8, T_9, T_{10}, T_{13}, T_{14}$.Thus

Table 2.12.Basis Function for first Node

12^{th}basis function	T_1	T_2	T_3	T_4
ϕ_{12}	$-x+y$	0	0	$-x+y$
T_5	T_6	T_7	T_8	T_9
0	0	0	0	0
T_{10}	T_{11}	T_{12}	T_{13}	T_{14}
0	$2-x-y$	$2-x-y$	0	0

2.2. Computing the Coefficients of Basis Functions
Consider the solution $U = \sum_{j=1}^{12} c_j \phi_j$.The coefficients c_j for the basis functions that correspond to boundary nodes$j = 5,6,7,8,9,10,11,12$; are chosen so that the solution satisfies the boundary conditions at those nodes.
To find the coefficients corresponding to the interior nodes to the interior nodes $i = 1,2,3,4$;we must minimize

$$\iint_R \{U_x^2 + U_y^2\} dxdy$$

For minimize $\frac{\partial U}{\partial c_i}$ for $i = 1,2,3,4$ (2.1)
Now, using the given Dirichlet boundary conditions, we obtain
$U(V_5) = U(V_6) = U(V_7) = U(V_{11}) = U(V_{12}) = 0,$and $U(V_8) = U(V_9) = U(V_{10}) = 1.$
Since we are looking for $U = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + c_4\phi_4 + \cdots \ldots \ldots \ldots + c_{12}\phi_{12}$;to satisfy the boundary conditions, and because each ϕ_j is zero except at node$j$,we must have

$$c_5 = c_6 = c_7 = c_{11} = c_{12} = 0 \text{ and } c_8 = c_9 = c_{10} = 1.$$

From equation (2.1), we get a system of linear equations

$$Ac = d \tag{2.2}$$

where, $\boldsymbol{A} = [a_{ij}], 1 \le i, j \le 4$.

$$a_{ij} = \iint_R \{[\phi_i]_x [\phi_j]_x + [\phi_i]_y [\phi_j]_y\} \, dxdy \tag{2.3}$$

$$\boldsymbol{c} = [c_i], \boldsymbol{d} = [d_i]; \; d_i = -\sum_{j=5}^{12} c_j b_{ij} \tag{2.4}$$

in which, for $1 \le i \le 4, 5 \le j \le 12$;

$$b_{ij} = \iint_R \{[\phi_i]_x [\phi_j]_x + [\phi_i]_y [\phi_j]_y\} \, dxdy. \tag{2.5}$$

Suppose $A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}, A_{16}, A_{17}$ and A_{18} are the areas of the triangles $T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9, T_{10}, T_{11}, T_{12}, T_{13}$ and T_{14} respectively. Therefore the area of triangle T_1 is

$$A_1 = 0.5 \left\| \begin{matrix} 1 & 1/2 & 1/2 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{matrix} \right\| = 0.25, \text{on simplification.}$$

We observe that the sizes of all triangles are same, hence

$$A_1 = A_2 = A_3 = A_4 = A_5 = A_6 = A_7 = A_8 = A_9 = A_{10} = A_{11} = A_{12} = A_{13} = A_{14} = 1/4.$$

For $i = 1, j = 1$,we get from (2.3)

$$\begin{aligned}
a_{11} &= \iint_R \{[\phi_1]_x [\phi_1]_x + [\phi_1]_y [\phi_1]_y\} \, dxdy \\
&= \iint_{T_1} \{[\phi_1]_x [\phi_1]_x + [\phi_1]_y [\phi_1]_y\} \, d \\
&+ \iint_{T_2} \{[\phi_1]_x [\phi_1]_x + [\phi_1]_y [\phi_1]_y\} \, dxdy + \cdots \\
&\ldots + \iint_{T_{13}} \{[\phi_1]_x [\phi_1]_x + [\phi_1]_y [\phi_1]_y\} \, dxdy \\
&+ \iint_{T_{14}} \{[\phi_1]_x [\phi_1]_x + [\phi_1]_y [\phi_1]_y\} \, dxdy \\
&= \iint_{T_1} \{4 + 0\} \, dxdy + \iint_{T_2} \{0 + 4\} \, dxdy \\
&+ \iint_{T_3} \{4 + 0\} \, dxdy + \iint_{T_4} \{0 + 4\} \, dxdy \\
&+0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 \\
&= 4 \iint_{T_1} dxdy + 4 \iint_{T_2} dxdy + 4 \iint_{T_3} dxdy + 4 \iint_{T_4} dxdy \\
&= 4 \times A_1 + 4 \times A_2 + 4 \times A_3 + 4 \times A_4 \\
&= 4 \times 0.25 + 4 \times 0.25 + 4 \times 0.25 + 4 \times 0.25 \\
&= 4.
\end{aligned}$$

For $i = 1, j = 2$,we get from (2.3)

$$\begin{aligned}
a_{12} &= \iint_R \{[\phi_1]_x [\phi_2]_x + [\phi_1]_y [\phi_2]_y\} \, dxdy \\
&= \iint_{T_1} \{[\phi_1]_x [\phi_2]_x + [\phi_1]_y [\phi_2]_y\} \, dxdy \\
&+ \iint_{T_2} \{[\phi_1]_x [\phi_2]_x + [\phi_1]_y [\phi_2]_y\} \, dxdy + \cdots \\
&\ldots + \iint_{T_{13}} \{[\phi_1]_x [\phi_2]_x + [\phi_1]_y [\phi_2]_y\} \, dxdy \\
&+ \iint_{T_{14}} \{[\phi_1]_x [\phi_2]_x + [\phi_1]_y [\phi_2]_y\} \, dxdy \\
&= 0 + 0 + \iint_{T_3} \{-2 + 0\} \, dxdy + \iint_{T_4} \{0 - 2\} \, dxdy + 0 \\
&+0 + 0 + +0 + 0 + 0 + +0 + 0 \\
&= -2 \iint_{T_3} dxdy - 2 \iint_{T_4} dxdy \\
&= -2 \times A_3 - 2 \times A_4 \\
&= -2 \times 0.25 - 2 \times 0.25 \\
&= -1.
\end{aligned}$$

For $i = 1, j = 3$,we get from (2.3)

$$a_{13} = \iint_R \{[\phi_1]_x[\phi_3]_x + [\phi_1]_y[\phi_3]_y\}\, dxdy$$
$$= \iint_{T_1} \{[\phi_1]_x[\phi_3]_x + [\phi_1]_y[\phi_3]_y\}\, dxdy$$
$$+ \iint_{T_2} \{[\phi_1]_x[\phi_3]_x + [\phi_1]_y[\phi_3]_y\}\, dxdy + \cdots$$
$$\ldots + \iint_{T_{13}} \{[\phi_1]_x[\phi_3]_x + [\phi_1]_y[\phi_3]_y\}\, dxdy$$
$$+ \iint_{T_{14}} \{[\phi_1]_x[\phi_3]_x + [\phi_1]_y[\phi_3]_y\}\, dxd$$
$$= 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$
$$+0$$
$$= 0.$$

For $i = 1, j = 4$,we get from (2.3)

$$a_{14} = \iint_R \{[\phi_1]_x[\phi_4]_x + [\phi_1]_y[\phi_4]_y\}\, dxdy$$
$$= \iint_{T_1} \{[\phi_1]_x[\phi_4]_x + [\phi_1]_y[\phi_4]_y\}\, dxdy$$
$$+ \iint_{T_2} \{[\phi_1]_x[\phi_4]_x + [\phi_1]_y[\phi_4]_y\}\, dxdy + \cdots$$
$$\ldots + \iint_{T_{13}} \{[\phi_1]_x[\phi_4]_x + [\phi_1]_y[\phi_4]_y\}\, dxdy$$
$$+ \iint_{T_{14}} \{[\phi_1]_x[\phi_4]_x + [\phi_1]_y[\phi_4]_y\}\, dxdy$$
$$= 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$
$$+0$$
$$= 0.$$

For $i = 2, j = 1$,we get from (2.3)

$$a_{21} = \iint_R \{[\phi_2]_x[\phi_1]_x + [\phi_2]_y[\phi_1]_y\}\, dxdy$$
$$= \iint_{T_1} \{[\phi_2]_x[\phi_1]_x + [\phi_2]_y[\phi_1]_y\}\, dxdy$$
$$+ \iint_{T_2} \{[\phi_2]_x[\phi_1]_x + [\phi_2]_y[\phi_1]_y\}\, dxdy + \cdots$$
$$\ldots + \iint_{T_{13}} \{[\phi_2]_x[\phi_1]_x + [\phi_2]_y[\phi_1]_y\}\, dxdy$$
$$+ \iint_{T_{14}} \{[\phi_2]_x[\phi_1]_x + [\phi_2]_y[\phi_1]_y\}\, dxdy$$
$$= 0 + 0 + \iint_{T_3} \{-2 + 0\}\, dxdy + \iint_{T_4} \{0 - 2\}\, dxdy + 0 + 0$$
$$+0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$
$$= -2 \iint_{T_3} dxdy - 2 \iint_{T_4} dxdy$$
$$= -2 \times A_3 - 2 \times A_4$$
$$= -2 \times 0.25 - 2 \times 0.25$$
$$= -1.$$

For $i = 2, j = 2$,we get from (2.3)

$$a_{22} = \iint_R \{[\phi_2]_x[\phi_2]_x + [\phi_2]_y[\phi_2]_y\}\, dxdy$$
$$= \iint_{T_1} \{[\phi_2]_x[\phi_2]_x + [\phi_2]_y[\phi_2]_y\}\, dxdy$$
$$+ \iint_{T_2} \{[\phi_2]_x[\phi_2]_x + [\phi_2]_y[\phi_2]_y\}\, dxdy + \cdots$$
$$\ldots + \iint_{T_{13}} \{[\phi_2]_x[\phi_2]_x + [\phi_2]_y[\phi_2]_y\}\, dxdy$$

$$+\iint_{T_{14}} \{[\phi_2]_x[\phi_2]_x + [\phi_2]_y[\phi_2]_y\}\, dxdy$$
$$= 0 + 0 + \iint_{T_3} \{1 + 1\}\, dxdy + \iint_{T_4} \{1 + 1\}\, dxdy$$
$$+\iint_{T_5} \{1 + 1\}\, dxdy + 0$$
$$+\iint_{T_8} \{1 + 1\}\, dxdy + \iint_{T_9} \{1 + 1\}\, dxdy$$
$$+\iint_{T_{10}} \{1 + 1\}\, dxdy + 0$$
$$+\iint_{T_{12}} \{1 + 1\}\, dxdy + \iint_{T_{13}} \{1 + 1\}\, dxdy + 0$$
$$= 2\iint_{T_3} dxdy + 2\iint_{T_4} dxdy + 2\iint_{T_5} dxdy$$
$$+2\iint_{T_8} dxdy + 2\iint_{T_9} dxdy + 2\iint_{T_{10}} dxdy + 2\iint_{T_{12}} dxdy + 2\iint_{T_{13}} dxdy$$
$$= 2 \times A_3 + 2 \times A_4 + 2 \times A_5 + 2 \times A_8 + 2 \times A_9 + 2 \times A_{10} + 2 \times A_{12} + 2 \times A_{13}$$
$$= 2 \times (0.25 + 0.25 + 0.25 + 0.25 + 0.25 + 0.25$$
$$+0.25 + 0.25)$$
$$= 4.$$

For $i = 2, j = 3$,we get from (2.3)

$$a_{23} = \iint_R \{[\phi_2]_x[\phi_3]_x + [\phi_2]_y[\phi_3]_y\}\, dxdy$$
$$= \iint_{T_1} \{[\phi_2]_x[\phi_3]_x + [\phi_2]_y[\phi_3]_y\}\, dxdy$$
$$+\iint_{T_2} \{[\phi_2]_x[\phi_3]_x + [\phi_2]_y[\phi_3]_y\}\, dxdy + \cdots$$
$$\ldots + \iint_{T_{13}} \{[\phi_2]_x[\phi_3]_x + [\phi_2]_y[\phi_3]_y\}\, dxdy$$
$$+\iint_{T_{14}} \{[\phi_2]_x[\phi_3]_x + [\phi_2]_y[\phi_3]_y\}\, dxdy$$
$$= 0 + 0 + \iint_{T_3} \{-2 + 0\}\, dxdy + \iint_{T_4} \{0 - 2\}\, dxdy$$
$$+\iint_{T_5} \{-2 + 0\}\, dxdy + 0 + 0 + \iint_{T_8} \{0 - 2\}\, dxdy + 0$$
$$+0 + 0 + 0 + 0 + 0$$
$$= -2\iint_{T_5} dxdy - 2\iint_{T_8} dxdy$$
$$= -2 \times A_5 - 2 \times A_8$$
$$= -2 \times 0.25 - 2 \times 0.25$$
$$= -1.$$

For $i = 2, j = 4$,we get from (2.3)

$$a_{24} = \iint_R \{[\phi_2]_x[\phi_4]_x + [\phi_2]_y[\phi_4]_y\}\, dxdy$$
$$= \iint_{T_1} \{[\phi_2]_x[\phi_4]_x + [\phi_2]_y[\phi_4]_y\}\, dxdy$$
$$+\iint_{T_2} \{[\phi_2]_x[\phi_4]_x + [\phi_2]_y[\phi_4]_y\}\, dxdy + \cdots$$
$$\ldots + \iint_{T_{13}} \{[\phi_2]_x[\phi_4]_x + [\phi_2]_y[\phi_4]_y\}\, dxdy$$
$$+\iint_{T_{14}} \{[\phi_2]_x[\phi_4]_x + [\phi_2]_y[\phi_4]_y\}\, dxdy$$
$$= 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$
$$+\iint_{T_{12}} \{0 - 2\}\, dxdy + \iint_{T_{13}} \{-2 + 0\}\, dxdy + 0$$
$$= -2\iint_{T_{12}} dxdy - 2\iint_{T_{13}} dxdy$$

$= -2 \times A_{12} - 2 \times A_{13}$
$= -2 \times 0.25 - 2 \times 0.25$
$= -1.$
For $i = 3, j = 1$,we get from (2.3)

$$a_{31} = \iint_R \{[\phi_3]_x[\phi_1]_x + [\phi_3]_y[\phi_1]_y\}\, dxdy$$
$$= \iint_{T_1} \{[\phi_3]_x[\phi_1]_x + [\phi_3]_y[\phi_1]_y\}\, dxdy$$
$$+ \iint_{T_2} \{[\phi_3]_x[\phi_1]_x + [\phi_3]_y[\phi_1]_y\}\, dxdy + \cdots$$
$$\ldots + \iint_{T_{13}} \{[\phi_3]_x[\phi_1]_x + [\phi_3]_y[\phi_1]_y\}\, dxdy$$
$$+ \iint_{T_{14}} \{[\phi_3]_x[\phi_1]_x + [\phi_3]_y[\phi_1]_y\}\, dxdy$$
$$= 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$
$$+0$$
$$= 0.$$

For $i = 3, j = 2$,we get from (2.3)

$$a_{32} = \iint_R \{[\phi_3]_x[\phi_2]_x + [\phi_3]_y[\phi_2]_y\}\, dxdy$$
$$= \iint_{T_1} \{[\phi_3]_x[\phi_2]_x + [\phi_3]_y[\phi_2]_y\}\, dxdy$$
$$+ \iint_{T_2} \{[\phi_3]_x[\phi_2]_x + [\phi_3]_y[\phi_2]_y\}\, dxdy + \cdots$$
$$\ldots + \iint_{T_{13}} \{[\phi_3]_x[\phi_2]_x + [\phi_3]_y[\phi_2]_y\}\, dxdy$$
$$+ \iint_{T_{14}} \{[\phi_3]_x[\phi_2]_x + [\phi_3]_y[\phi_2]_y\}\, dxdy$$
$$= 0 + 0 + \iint_{T_3} \{-2 + 0\}\, dxdy + \iint_{T_4} \{0 - 2\}\, dxdy + 0$$
$$+0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$
$$= -2 \iint_{T_3} dxdy - 2 \iint_{T_4} dxdy$$
$$= -2 \times A_3 - 2 \times A_4$$
$$= -2 \times 0.25 - 2 \times 0.25$$
$$= -1.$$

For $i = 3, j = 3$,we get from (2.3)

$$a_{33} = \iint_R \{[\phi_3]_x[\phi_3]_x + [\phi_3]_y[\phi_3]_y\}\, dxdy$$
$$= \iint_{T_1} \{[\phi_3]_x[\phi_3]_x + [\phi_3]_y[\phi_3]_y\}\, dxdy$$
$$+ \iint_{T_2} \{[\phi_3]_x[\phi_3]_x + [\phi_3]_y[\phi_3]_y\}\, dxdy + \cdots$$
$$\ldots + \iint_{T_{13}} \{[\phi_3]_x[\phi_3]_x + [\phi_3]_y[\phi_3]_y\}\, dxdy$$
$$+ \iint_{T_{14}} \{[\phi_3]_x[\phi_3]_x + [\phi_3]_y[\phi_3]_y\}\, dxdy$$
$$= 0 + 0 + 0 + 0 + \iint_{T_5} \{4 + 0\}\, dxdy$$
$$+ \iint_{T_6} \{0 + 4\}\, dxdy + \iint_{T_7} \{4 + 0\}\, dxdy$$
$$+ \iint_{T_8} \{0 + 4\}\, dxdy + 0 + 0 + 0 + 0 + 0 + 0$$
$$= 4 \iint_{T_5} dxdy + 4 \iint_{T_6} dxdy + 4 \iint_{T_7} dxdy + 4 \iint_{T_8} dxdy$$
$$= 4 \times A_5 + 4 \times A_6 + 4 \times A_7 + 4 \times A_8$$
$$= 4 \times 0.25 + 4 \times 0.25 + 4 \times 0.25 + 4 \times 0.25$$

$= 4.$

For $i = 3, j = 4$,we get from (2.3)

$$a_{34} = \iint_R \{[\phi_3]_x[\phi_4]_x + [\phi_3]_y[\phi_4]_y\}\, dxdy$$
$$= \iint_{T_1} \{[\phi_3]_x[\phi_4]_x + [\phi_3]_y[\phi_4]_y\}\, dxdy$$
$$+ \iint_{T_2} \{[\phi_3]_x[\phi_4]_x + [\phi_3]_y[\phi_4]_y\}\, dxdy + \cdots$$
$$\ldots + \iint_{T_{13}} \{[\phi_3]_x[\phi_4]_x + [\phi_3]_y[\phi_4]_y\}\, dxdy$$
$$+ \iint_{T_{14}} \{[\phi_3]_x[\phi_4]_x + [\phi_3]_y[\phi_4]_y\}\, dxdy$$
$$= 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$
$$+0$$
$$= 0.$$

For $i = 4, j = 1$,we get from (2.3)

$$a_{41} = \iint_R \{[\phi_4]_x[\phi_1]_x + [\phi_4]_y[\phi_1]_y\}\, dxdy$$
$$= \iint_{T_1} \{[\phi_4]_x[\phi_1]_x + [\phi_4]_y[\phi_1]_y\}\, dxdy$$
$$+ \iint_{T_2} \{[\phi_4]_x[\phi_1]_x + [\phi_4]_y[\phi_1]_y\}\, dxdy + \cdots$$
$$\ldots + \iint_{T_{13}} \{[\phi_4]_x[\phi_1]_x + [\phi_4]_y[\phi_1]_y\}\, dxdy$$
$$+ \iint_{T_{14}} \{[\phi_4]_x[\phi_1]_x + [\phi_4]_y[\phi_1]_y\}\, dxdy$$
$$= 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$
$$+0$$
$$= 0.$$

For $i = 4, j = 2$,we get from (2.3)

$$a_{42} = \iint_R \{[\phi_4]_x[\phi_2]_x + [\phi_4]_y[\phi_2]_y\}\, dxdy$$
$$= \iint_{T_1} \{[\phi_4]_x[\phi_2]_x + [\phi_4]_y[\phi_2]_y\}\, dxdy$$
$$+ \iint_{T_2} \{[\phi_4]_x[\phi_2]_x + [\phi_4]_y[\phi_2]_y\}\, dxdy + \cdots$$
$$\ldots + \iint_{T_{13}} \{[\phi_4]_x[\phi_2]_x + [\phi_4]_y[\phi_2]_y\}\, dxdy$$
$$+ \iint_{T_{14}} \{[\phi_4]_x[\phi_2]_x + [\phi_4]_y[\phi_2]_y\}\, dxdy$$
$$= 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$
$$+ \iint_{T_{12}} \{-2 + 0\}\, dxdy + \iint_{T_{13}} \{0 - 2\}\, dxdy + 0$$
$$= -2 \iint_{T_{12}} dxdy - 2 \iint_{T_{13}} dxdy$$
$$= -2 \times A_{12} - 2 \times A_{13}$$
$$= -2 \times 0.25 - 2 \times 0.25$$
$$= -1.$$

For $i = 4, j = 3$,we get from (2.3)

$$a_{43} = \iint_R \{[\phi_4]_x[\phi_3]_x + [\phi_4]_y[\phi_3]_y\}\, dxdy$$
$$= \iint_{T_1} \{[\phi_4]_x[\phi_3]_x + [\phi_4]_y[\phi_3]_y\}\, dxdy$$
$$+ \iint_{T_2} \{[\phi_4]_x[\phi_3]_x + [\phi_4]_y[\phi_3]_y\}\, dxdy + \cdots$$

$$\dots + \iint_{T_{13}} \{[\phi_4]_x[\phi_3]_x + [\phi_4]_y[\phi_3]_y\}\, dxdy$$
$$+ \iint_{T_{14}} \{[\phi_4]_x[\phi_3]_x + [\phi_4]_y[\phi_3]_y\}\, dxdy$$
$$= 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$
$$+0$$
$$= 0.$$

For $i = 4, j = 4$,we get from (2.3)

$$a_{44} = \iint_R \{[\phi_4]_x[\phi_4]_x + [\phi_4]_y[\phi_4]_y\}\, dxdy$$
$$= \iint_{T_1} \{[\phi_4]_x[\phi_4]_x + [\phi_4]_y[\phi_4]_y\}\, dxdy$$
$$+ \iint_{T_2} \{[\phi_4]_x[\phi_4]_x + [\phi_4]_y[\phi_4]_y\}\, dxdy + \cdots$$
$$\dots + \iint_{T_{13}} \{[\phi_4]_x[\phi_4]_x + [\phi_4]_y[\phi_4]_y\}\, dxdy$$
$$+ \iint_{T_{14}} \{[\phi_4]_x[\phi_4]_x + [\phi_4]_y[\phi_4]_y\}\, dxdy$$
$$= 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$
$$+ \iint_{T_{11}} \{4 + 0\}\, dxdy + \iint_{T_{12}} \{0 + 4\}\, dxdy$$
$$+ \iint_{T_{13}} \{4 + 0\}\, dxdy + \iint_{T_{14}} \{0 + 4\}\, dxdy$$
$$= 4 \iint_{T_{11}} dxdy + 4 \iint_{T_{12}} dxdy + 4 \iint_{T_{13}} dxdy + 4 \iint_{T_{14}} dxdy$$
$$= 4 \times A_{11} + 4 \times A_{12} + 4 \times A_{13} + 4 \times A_{14}$$
$$= 4 \times 0.25 + 4 \times 0.25 + 4 \times 0.25 + 4 \times 0.25$$
$$= 4.$$

For $i = 1$,we get from (2.4)

$$d_1 = -\sum_{j=5}^{12} c_j b_{1j} = -(c_5 b_{15} + c_6 b_{16} + c_7 b_{17} + c_8 b_{18} + c_9 b_{19} + c_{10} b_{110} + c_{11} b_{111} + c_{12} b_{112}). \qquad (2.6)$$

Since $c_5 = c_6 = c_7 = c_{11} = c_{12} = 0$,so we only calculate the coefficients for b_{18}, b_{19}, b_{110}.

For $i = 1, j = 8$, we get from (2.5)

$$b_{18} = \iint_R \{[\phi_1]_x[\phi_8]_x + [\phi_1]_y[\phi_8]_y\}\, dxdy$$
$$= \iint_{T_1} \{[\phi_1]_x[\phi_8]_x + [\phi_1]_y[\phi_8]_y\}\, dxdy$$
$$+ \iint_{T_2} \{[\phi_1]_x[\phi_8]_x + [\phi_1]_y[\phi_8]_y\}\, dxdy + \cdots$$
$$\dots + \iint_{T_{13}} \{[\phi_1]_x[\phi_8]_x + [\phi_1]_y[\phi_8]_y\}\, dxdy$$
$$+ \iint_{T_{14}} \{[\phi_1]_x[\phi_8]_x + [\phi_1]_y[\phi_8]_y\}\, dxdy$$
$$= 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$
$$+0$$
$$= 0.$$

For $i = 1, j = 9$, we get from (2.5)

$$b_{19} = \iint_R \{[\phi_1]_x[\phi_9]_x + [\phi_1]_y[\phi_9]_y\}\, dxdy$$
$$= \iint_{T_1} \{[\phi_1]_x[\phi_9]_x + [\phi_1]_y[\phi_9]_y\}\, dxdy$$
$$+ \iint_{T_2} \{[\phi_1]_x[\phi_9]_x + [\phi_1]_y[\phi_9]_y\}\, dxdy + \cdots$$
$$\dots + \iint_{T_{13}} \{[\phi_1]_x[\phi_9]_x + [\phi_1]_y[\phi_9]_y\}\, dxdy$$

$$+\iint_{T_{14}}\{[\phi_1]_x[\phi_9]_x+[\phi_1]_y[\phi_9]_y\}\,dxdy$$
$$=0+0+0+0+0+0+0+0+0+0+0+0+0$$
$$+0$$
$$=0.$$

For $i=1, j=10$, we get from (2.5)

$$b_{110}=\iint_R\{[\phi_1]_x[\phi_{10}]_x+[\phi_1]_y[\phi_{10}]_y\}\,dxdy$$
$$=\iint_{T_1}\{[\phi_1]_x[\phi_{10}]_x+[\phi_1]_y[\phi_{10}]_y\}\,dxdy$$
$$+\iint_{T_2}\{[\phi_1]_x[\phi_{10}]_x+[\phi_1]_y[\phi_{10}]_y\}\,dxdy+\cdots$$
$$\ldots+\iint_{T_{13}}\{[\phi_1]_x[\phi_{10}]_x+[\phi_1]_y[\phi_{10}]_y\}\,dxdy$$
$$+\iint_{T_{14}}\{[\phi_1]_x[\phi_{10}]_x+[\phi_1]_y[\phi_{10}]_y\}\,dxdy$$
$$=0+0+0+0+0+0+0+0+0+0+0+0+0$$
$$+0$$
$$=0.$$

Hence the equation (2.6) becomes

$d_1=0.$

For $i=2$, we get from (2.4)

$$d_2=-\sum_{j=5}^{12}c_j b_{2j}=-(c_5b_{25}+c_6b_{26}+c_7b_{27}+c_8b_{28}+c_9b_{29}+c_{10}b_{210}+c_{11}b_{211}+c_{12}b_{212}). \tag{2.7}$$

Since $c_5=c_6=c_7=c_{11}=c_{12}=0$, so we only calculate the coefficients for b_{28}, b_{29}, b_{210}

For $i=2, j=8$, we get from (2.5)

$$b_{28}=\iint_R\{[\phi_2]_x[\phi_8]_x+[\phi_2]_y[\phi_8]_y\}\,dxdy$$
$$=\iint_{T_1}\{[\phi_2]_x[\phi_8]_x+[\phi_2]_y[\phi_8]_y\}\,dxdy$$
$$+\iint_{T_2}\{[\phi_2]_x[\phi_8]_x+[\phi_2]_y[\phi_8]_y\}\,dxdy+\cdots$$
$$\ldots+\iint_{T_{13}}\{[\phi_2]_x[\phi_8]_x+[\phi_2]_y[\phi_8]_y\}\,dxdy$$
$$+\iint_{T_{14}}\{[\phi_2]_x[\phi_8]_x+[\phi_2]_y[\phi_8]_y\}\,dxdy$$
$$=0+0+0+0+0+0+0+0+0+0+0+0+0$$
$$+0$$
$$=0.$$

For $i=2, j=9$,we get from (2.5)

$$b_{29}=\iint_R\{[\phi_2]_x[\phi_9]_x+[\phi_2]_y[\phi_9]_y\}\,dxdy$$
$$=\iint_{T_1}\{[\phi_2]_x[\phi_9]_x+[\phi_2]_y[\phi_9]_y\}\,dxdy$$
$$+\iint_{T_2}\{[\phi_2]_x[\phi_9]_x+[\phi_2]_y[\phi_9]_y\}\,dxdy+\cdots$$
$$\ldots+\iint_{T_{13}}\{[\phi_2]_x[\phi_9]_x+[\phi_2]_y[\phi_9]_y\}\,dxdy$$
$$+\iint_{T_{14}}\{[\phi_2]_x[\phi_9]_x+[\phi_2]_y[\phi_9]_y\}\,dxdy$$
$$=0+0+0+0+0+0+0+0+\iint_{T_9}\{0-2\}\,dxdy$$
$$+\iint_{T_{10}}\{-2+0\}\,dxdy+0+0+0+0$$
$$=-2\iint_{T_9}dxdy-2\iint_{T_{10}}dxdy$$

$= -2 \times A_9 - 2 \times A_{10}$
$= -2 \times 0.25 - 2 \times 0.25$
$= -1.$
For $i = 2, j = 10$, we get from (2.5)
$$b_{210} = \iint_R \{[\phi_2]_x[\phi_{10}]_x + [\phi_2]_y[\phi_{10}]_y\} \, dxdy$$
$$= \iint_{T_1} \{[\phi_2]_x[\phi_{10}]_x + [\phi_2]_y[\phi_{10}]_y\} \, dxdy$$
$$+ \iint_{T_2} \{[\phi_2]_x[\phi_{10}]_x + [\phi_2]_y[\phi_{10}]_y\} \, dxdy + \cdots$$
$$\ldots + \iint_{T_{13}} \{[\phi_2]_x[\phi_{10}]_x + [\phi_2]_y[\phi_{10}]_y\} \, dxdy$$
$$+ \iint_{T_{14}} \{[\phi_2]_x[\phi_{10}]_x + [\phi_2]_y[\phi_{10}]_y\} \, dxdy$$
$$= 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$
$$+0$$
$$= 0.$$
Hence the equation (2.7) becomes
$d_2 = 1.$
For $i = 3$,we get from (2.4)
$$d_3 = -\sum_{j=5}^{12} c_j b_{3j} = -(c_5 b_{35} + c_6 b_{36} + c_7 b_{37} + c_8 b_{38} + c_9 b_{39} + c_{10} b_{310} + c_{11} b_{311} + c_{12} b_{312}) \quad (2.8)$$
Since $c_5 = c_6 = c_7 = c_{11} = c_{12} = 0$,so we only calculate the coefficients for b_{38}, b_{39}, b_{310}.
For $i = 3, j = 8$,we get from (2.5)
$$b_{38} = \iint_R \{[\phi_3]_x[\phi_8]_x + [\phi_3]_y[\phi_8]_y\} \, dxdy$$
$$= \iint_{T_1} \{[\phi_3]_x[\phi_8]_x + [\phi_3]_y[\phi_8]_y\} \, dxdy$$
$$+ \iint_{T_2} \{[\phi_3]_x[\phi_8]_x + [\phi_3]_y[\phi_8]_y\} \, dxdy + \cdots$$
$$\ldots + \iint_{T_{13}} \{[\phi_3]_x[\phi_8]_x + [\phi_3]_y[\phi_8]_y\} \, dxdy$$
$$+ \iint_{T_{14}} \{[\phi_3]_x[\phi_8]_x + [\phi_3]_y[\phi_8]_y\} \, dxdy$$
$$= 0 + 0 + 0 + 0 + 0 + 0 + \iint_{T_7} \{-2 + 0\} \, dxdy$$
$$+ \iint_{T_8} \{0 - 2\} \, dxdy + 0 + 0 + 0 + 0 + 0 + 0$$
$$= -2 \iint_{T_7} \quad dxdy - 2 \iint_{T_8} \quad dxdy$$
$$= -2 \times A_7 - 2 \times A_8$$
$= -2 \times 0.25 - 2 \times 0.25$
$= -1.$
For $i = 3, j = 9$, we get from (2.5)
$$b_{39} = \iint_R \{[\phi_3]_x[\phi_9]_x + [\phi_3]_y[\phi_9]_y\} \, dxdy$$
$$= \iint_{T_1} \{[\phi_3]_x[\phi_9]_x + [\phi_3]_y[\phi_9]_y\} \, dxdy$$
$$+ \iint_{T_2} \{[\phi_3]_x[\phi_9]_x + [\phi_3]_y[\phi_9]_y\} \, dxdy + \cdots$$
$$\ldots + \iint_{T_{13}} \{[\phi_3]_x[\phi_9]_x + [\phi_3]_y[\phi_9]_y\} \, dxdy$$
$$+ \iint_{T_{14}} \{[\phi_3]_x[\phi_9]_x + [\phi_3]_y[\phi_9]_y\} \, dxdy$$
$$= 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$
$$+0$$

$= 0.$

For $i = 3, j = 10$, we get from (2.5)

$$b_{310} = \iint_R \{[\phi_3]_x[\phi_{10}]_x + [\phi_3]_y[\phi_{10}]_y\}\,dxdy$$
$$= \iint_{T_1} \{[\phi_3]_x[\phi_{10}]_x + [\phi_3]_y[\phi_{10}]_y\}\,dxdy$$
$$+ \iint_{T_2} \{[\phi_3]_x[\phi_{10}]_x + [\phi_3]_y[\phi_{10}]_y\}\,dxdy + \cdots$$
$$\ldots + \iint_{T_{13}} \{[\phi_3]_x[\phi_{10}]_x + [\phi_3]_y[\phi_{10}]_y\}\,dxdy$$
$$+ \iint_{T_{14}} \{[\phi_3]_x[\phi_{10}]_x + [\phi_3]_y[\phi_{10}]_y\}\,dxdy$$
$$= 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$
$$+0$$
$$= 0.$$

Hence the equation (2.8) becomes

$d_3 = 1.$

For $i = 4$,we get from (2.4)

$$d_4 = -\sum_{j=5}^{12} c_j b_{4j} = -(c_5 b_{45} + c_6 b_{46} + c_7 b_{47} + c_8 b_{48} + c_9 b_{49} + c_{10} b_{410} + c_{11} b_{411} + c_{12} b_{412}). \quad (2.9)$$

Since $c_5 = c_6 = c_7 = c_{11} = c_{12} = 0$,so we only calculate the coefficients for b_{48}, b_{49}, b_{410}.

For $i = 4, j = 8$, we get from (2.5)

$$b_{48} = \iint_R \{[\phi_4]_x[\phi_8]_x + [\phi_4]_y[\phi_8]_y\}\,dxdy$$
$$= \iint_{T_1} \{[\phi_4]_x[\phi_8]_x + [\phi_4]_y[\phi_8]_y\}\,dxdy$$
$$+ \iint_{T_2} \{[\phi_4]_x[\phi_8]_x + [\phi_4]_y[\phi_8]_y\}\,dxdy + \cdots$$
$$\ldots + \iint_{T_{13}} \{[\phi_4]_x[\phi_8]_x + [\phi_4]_y[\phi_8]_y\}\,dxdy$$
$$+ \iint_{T_{14}} \{[\phi_4]_x[\phi_8]_x + [\phi_4]_y[\phi_8]_y\}\,dxdy$$
$$= 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$
$$+0$$
$$= 0.$$

For $i = 4, j = 9$, we get from (2.5)

$$b_{49} = \iint_R \{[\phi_4]_x[\phi_9]_x + [\phi_4]_y[\phi_9]_y\}\,dxdy$$
$$= \iint_{T_1} \{[\phi_4]_x[\phi_9]_x + [\phi_4]_y[\phi_9]_y\}\,dxdy$$
$$+ \iint_{T_2} \{[\phi_4]_x[\phi_9]_x + [\phi_4]_y[\phi_9]_y\}\,dxdy + \cdots$$
$$\ldots + \iint_{T_{13}} \{[\phi_4]_x[\phi_9]_x + [\phi_4]_y[\phi_9]_y\}\,dxdy$$
$$+ \iint_{T_{14}} \{[\phi_4]_x[\phi_9]_x + [\phi_4]_y[\phi_9]_y\}\,dxdy$$
$$= 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$$
$$+0$$
$$= 0.$$

For $i = 4, j = 10$,we get from (2.5)

$$b_{410} = \iint_R \{[\phi_4]_x[\phi_{10}]_x + [\phi_4]_y[\phi_{10}]_y\}\,dxdy$$
$$= \iint_{T_1} \{[\phi_4]_x[\phi_{10}]_x + [\phi_4]_y[\phi_{10}]_y\}\,dxdy$$

$$+\iint_{T_2}\{[\phi_4]_x[\phi_{10}]_x+[\phi_4]_y[\phi_{10}]_y\}\,dxdy+\cdots$$
$$\ldots+\iint_{T_{13}}\{[\phi_4]_x[\phi_{10}]_x+[\phi_4]_y[\phi_{10}]_y\}\,dxdy$$
$$+\iint_{T_{14}}\{[\phi_4]_x[\phi_{10}]_x+[\phi_4]_y[\phi_{10}]_y\}\,dxdy$$
$$=0+0+0+0+0+0+0+0+0+0+0+0$$
$$+\iint_{T_{13}}\{-2+0\}\,dxdy+\iint_{T_{14}}\{0-2\}\,dxdy$$
$$=-2\iint_{T_{13}}\ dxdy-2\iint_{T_{14}}\ dxdy$$
$$=-2\times A_{13}-2\times A_{14}$$
$=-2\times 0.25-2\times 0.25$
$=-1.$
Hence the equation (2.9) becomes
$d_4=1.$

Now putting the values of
$a_{11}, a_{12}, a_{13}, a_{14}, a_{21}, a_{22}, a_{23}, a_{24}, a_{31}, a_{32}, a_{33}, a_{34}, a_{41}, a_{42}$,
a_{43}, a_{44}, andd_1, d_2, d_3, d_4;in linear system (2.2), then we obtain the following equations
$4c_1-c_2=0$
$-c_1+4c_2-c_3-c_4=1$
$-c_2+4c_3=1$
$-c_2+4c_4=1$
Solving these we get, $c_1=0.1154, c_2=0.4615, c_3=0.3654, c_4=0.3654$ (Approximation).

2.3. The solution of the potential function
Therefore, the approximate solution of the potential function is
$U=0.1154\phi_1+0.4615\phi_2+0.3654\phi_3+0.3654\phi_4+\phi_8+\phi_9+\phi_{10}$.
This simplifies to

$$U=\begin{cases} 0.2308x & \text{on } T_1 \\ 0.2308y & \text{on } T_2 \\ -0.2308+0.2308x+0.4615y & \text{on } T_3 \\ -0.2308+0.4615x+0.2308y & \text{on } T_4 \\ -0.2692+0.2692x+0.4615y & \text{on } T_5 \\ 0.7308y & \text{on } T_6 \\ -0.5385+0.2692x+y & \text{on } T_7 \\ -0.8077+0.5385x+0.7308y & \text{on } T_8 \\ -0.6154+0.5385x+0.5385y & \text{on } T_9 \\ -0.6154+0.5385x+0.5385y & \text{on } T_{10} \\ 0.7308x & \text{on } T_{11} \\ -0.2692+0.4615x+0.2692y & \text{on } T_{12} \\ -0.8077+0.7308x+0.5385y & \text{on } T_{13} \\ -0.5385+x+0.2692y & \text{on } T_{14} \end{cases}$$

2.4. Finite Element MATLAB Program for the solution of the potential function
APPENDIX X:
X.1. MATLAB program for 2D Laplace equation using finite element method.
Result:

```
U =
0    0.2308    0
0    0    0.2308
-0.2308   0.2308   0.4615
-0.2308   0.4615   0.2308
-0.2692   0.2692   0.4615
0    0   0.7308
-0.5385   0.2692   1.0000
-0.8077   0.5385   0.7308
-0.6154   0.5385   0.5385
```

-0.6154   0.5385   0.5385
0   0.7308   0
-0.2692   0.4615   0.2692
-0.8077   0.7308   0.5385
-0.5385   1.0000   0.2692

This tell us that the final solution is

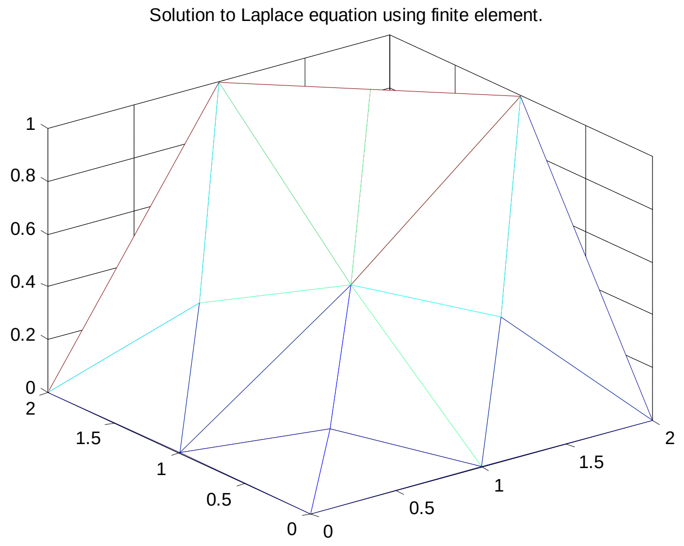
$$U = \begin{cases} 0 + 0.2308x + 0y & \text{on } T_1 \\ 0 + 0x + 0.2308y & \text{on } T_2 \\ -0.2308 + 0.2308x + 0.4615y & \text{on } T_3 \\ -0.2308 + 0.4615x + 0.2308y & \text{on } T_4 \\ -0.2692 + 0.2692x + 0.4615y & \text{on } T_5 \\ 0 + 0x + 0.7308y & \text{on } T_6 \\ -0.5385 + 0.2692x + y & \text{on } T_7 \\ -0.8077 + 0.5385x + 0.7308y & \text{on } T_8 \\ -0.6154 + 0.5385x + 0.5385y & \text{on } T_9 \\ -0.6154 + 0.5385x + 0.5385y & \text{on } T_{10} \\ 0 + 0.7308x + 0y & \text{on } T_{11} \\ -0.2692 + 0.4615x + 0.2692y & \text{on } T_{12} \\ -0.8077 + 0.7308x + 0.5385y & \text{on } T_{13} \\ -0.5385 + x + 0.2692y & \text{on } T_{14} \end{cases}$$

The computed values at the nodes, *U(x, y)* can be found from these formulas, as shown in the following table:

Table 2.13. the values of the potential function at the nodes

Node	**X**	**Y**	**U**	Results from formula
1	$\frac{1}{2}$	$\frac{1}{2}$	0.1154	$T_1, T_2, T_3, or T_4$
2	**1**	**1**	0.4615	$T_3, T_4, T_5, T_8, T_9, T_{10}, T_{12}, or T_{13}$
3	$\frac{3}{2}$	$\frac{1}{2}$	0.3654	$T_5, T_6, T_7, or T_8$
4	$\frac{1}{2}$	$\frac{3}{2}$	0.3654	$T_{11}, T_{12}, T_{13}, or T_{14}$
5	**0**	**0**	**0**	$T_1 or T_2$
6	**1**	**0**	**0**	$T_2, T_3, T_5, or T_6$
7	**2**	**0**	**0**	$T_6, or T_7$
8	**2**	**1**	**1**	$T_7, T_8, or T_9$
9	$\frac{3}{2}$	$\frac{3}{2}$	**1**	$T_9 or T_{10}$
10	**1**	**2**	**1**	$T_{10}, T_{13}, or T_{14}$
11	**0**	**2**	**0**	$T_{11} or T_{14}$
12	**0**	**1**	**0**	$T_1, T_4, T_{11}, or T_{12}$

2.5. Finite Element MATLAB Program for plotting the solution PPENDIX X:
X.2. MATLAB program for plotting solution using finite element method.



III. CONCLUSION

Our aim is to introduce the Finite-Element method for elliptic partial differential equations and particularly to solve two dimensional Laplace equations with Dirichlet boundary conditions. The Finite-Element method was chosen as an elliptic partial differential solver and so it fundamental was discussed. Then the Finite-Element method was used to solve two dimensional Laplace equations with Dirichlet boundary conditions for irregular shape. Then the accuracy of the developed scheme was shown on the basis of the numbers of the elements. The Finite-Element method is flexibly applied to elliptic partial differential equations and it can also be applied to parabolic and hyperbolic partial differential equations, but the minimization procedure is more difficult. Many physical problems have boundary conditions involving derivatives and irregularly shaped boundaries. The Finite-Element method includes the boundary conditions as integrals in a functional that is being minimized, so the construction procedure is independent of the particular boundary conditions of the problem.

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Authors have made equal contributions for paper.

Competing Interests
The authors declare that they have no competing interests.

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