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Mathematical Analysis Of Reaction Diffusion Equation With Ecological Parameters

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Abstract: This work is associated with the solution of reaction diffusion equation with ecological parameters. The mathematical analysis of reaction diffusion equation with ecological parameters has been conducted by the mixed use of inequalities and integral transform methods. The problem is solved in the form of power series to obtain approximate solution with simply reckonable terms and the exact solutions can be accomplished by the mean of acknowledged structure of the series solutions. To show the effectiveness of method, the examples are also illustrated.

Keywords: Reaction diffusion equation, ecological parameters, integral transform, Holder Inequality.

1. INTRODUCTION

During the end of 20thcentury, much interest is taking place in the study of reaction diffusion equation. Some mathematicians had started this work, but in the next decade it is used by many of researchers from different streams. The researchers from the different fields such as biology, environmental science, chemical science etc. are also using it for purpose to their actual need and applications. Now it is extended to more regions. Here we have also extended the reaction diffusion equation to reaction diffusion equation with ecological parameters and its solution will be useful in the area of life science especially in environmental science and fluid dynamics. Its application is not limited to mathematics only, but it can be widely used by researchers from the field of biology, geology, physics and ecology. It could be used in eco environment system like treating water impurities and purify the air in the atmosphere.

The literature review in this field shows that it is the area where researchers from different area are doing work from last three decade. It also explains the numerous applications of reaction diffusion equation. Grindord [1] explains the theory and application of reaction diffusion equation. Murray [2, 3] expresses the application of reaction diffusion in mathematical biology. Lesnic [4] proposes the decomposition method for Cauchy advection diffusion problems. Aminataei et al [5] highlighted the comparison of stability of Adomian decomposition method for solving the diffusion convection reaction equation. Lesnic [6] again pointed out the application of the decomposition method for Cauchy reaction diffusion problems. Bataieneh et al [7] give the homotopy analysis method for Cauchy reaction diffusion problems. Yildirim [8] focused on the application of He's Homotopy perturbation method for solving the Cauchy reaction diffusion problem. Kumar et al [9, 10] mentioned the solution of reaction diffusion equations by using Homotopy perturbation method, they also explained a mathematical model to solve reaction diffusion equation by using homotopy perturbation method and Adomian decomposition method. Wu [11] extend the Adomian decomposition method for non smooth initial values problems. Bhadauria et al [12, 13] formulated a mathematical model to solve reaction diffusion equation by differential transform method; they also give the solution of reaction diffusion equation by Adomian decomposition method. Maurya et al [14] mentioned the analytical solution of a new approach to reaction diffusion equation by NHPM. Li et al [15] developed reaction diffusion system prediction based on convolution neural network. On the base of above literature review we can conclude that there are many methods available which explained the solution of reaction diffusion equation. In these methods the homotopy analysis method is used by most of the authors, Adomian decomposition method, differential transform method, and NHPM method is used by some researchers. In this paper, we proposed an integral transform method to solve newly generated partial differential equation namely reaction diffusion equation with ecological parameters. Here we are also use the holder inequality. To indicate its application, we also illustrate it will the help of examples.

2. DESCRIPTION OF PROBLEM AND ITS SOLUTION

Reaction diffusion equation with ecological parameters is given by-

$$u_t = Du_{xx} - Ru\left(\frac{u}{N} - 1\right) \tag{1}$$

under the boundary condition

$$\left.\begin{array}{ll} \text{(i)} & u(0, t) = 0 \\ \text{(ii)} & u(\pi, t) = 0 \\ \text{(iii)} & u(x, 0) = u_0(x) \end{array}\right\} \tag{2}$$

where D is diffusion coefficient

R is the linear reproduction rate
and N is ecological parameter which carries the capacity of environment.

By equation (1), we have

$$u_t = Du_{xx} + Ru - \frac{R}{N}u^2 \tag{3}$$

Multiplying above equation by sin nx we get

$$u_t \sin nx = Du_{xx} \sin nx + Ru \sin nx - \frac{R}{N}u^2 \sin nx \tag{4}$$

Integrating above with respect to x under the limit 0 to π

$$\int_0^\pi u_t \sin nx\, dx = D\int_0^\pi u_{xx} \sin nx\, dx + R\int_0^\pi u \sin nx\, dx - \frac{R}{N}\int_0^\pi u^2 \sin nx\, dx \tag{5}$$

Multiplying above by $\frac{2}{\pi}$ on both sides

$$\frac{2}{\pi}\int_0^\pi u_t \sin nx\, dx = \frac{2D}{\pi}\int_0^\pi u_{xx} \sin nx\, dx + \frac{2R}{\pi}\int_0^\pi u \sin nx\, dx - \frac{2R}{\pi N}\int_0^\pi u^2 \sin nx\, dx \tag{6}$$

$$\frac{\partial}{\partial t}\left[\frac{2}{\pi}\int_0^\pi u \sin nx\, dx\right] = \frac{2D}{\pi}\left[\left(\sin nx.u_x\right)_0^\pi - n\int_0^\pi u_x \cos nx\, dx\right] + R\left[\frac{2}{\pi}\int_0^\pi u \sin nx\right] - \frac{2R}{\pi N}\int_0^\pi u^2 \sin nx\, dx \tag{7}$$

Let $f = f(t) = \frac{2}{\pi}\int_0^\pi u \sin nx\, dx$ (8)

By (7) and (8)

$$\frac{\partial f}{\partial t} = \frac{2D}{\pi}\left[0 - n\left[u \cos nx\right]_0^\pi + n\int_0^\pi (-n) u \sin nx\, dx\right] + Rf - \frac{2R}{N\pi}\int_0^\pi u^2 \sin nx\, dx \tag{9}$$

$$\frac{\partial f}{\partial t} = \left(R - n^2 D\right) f - \frac{2R}{\pi N}\int_0^\pi u^2 \sin nx\, dx \tag{10}$$

By Holder's Inequality

$$\int_a^b |fg|\, dx \le \left[\int_a^b |f(x)|^p\, dx\right]^{1/p} \left[\int_a^b |g(x)|^{\frac{p}{p-1}}\, dx\right]^{1-\frac{1}{p}} \tag{11}$$

$$\int_0^\pi u \sin nx\, dx \le \left[\int_0^\pi u^p \sin nx\, dx\right]^{\frac{1}{p}} \left[\int_0^\pi \sin nx\, dx\right]^{1-\frac{1}{p}} \tag{12}$$

Taking p = 2 and squaring both sides in the above equation

$$\left[\int_0^\pi u \sin nx\, dx\right]^2 \le \left[\int_0^\pi u^2 \sin nx\, dx\right]\left[\int_0^\pi \sin nx\, dx\right] \tag{13}$$

Using (8) in (13), it is obtained as

$$f^2 \le \frac{4}{\pi^2}\left[\int_0^{\pi} u^2 \sin nx\, dx\right]\left[\int_0^{\pi} \sin nx\, dx\right] \tag{14}$$

which may be expressed with strict sign as

$$\int_0^{\pi} u^2 \sin nx\, dx = \frac{n\pi^2 f^2}{4(1-\cos n\pi)} \tag{15}$$

Substituting (15) in (10), we obtain

$$\frac{\partial f}{\partial t} = Af - Bf^2 \tag{16}$$

where $A = R - n^2 D$ and $B = \frac{n\pi R}{4N}$

Dividing above equation by $-1/f^2$

$$-\frac{1}{f^2}\frac{\partial f}{\partial t} + \frac{A}{f} = B \tag{17}$$

whose solution is obtained as

$$f = f(t) = \frac{Ae^{At}}{Be^{At} + CA} \tag{18}$$

By inverse transform

$$u(x,t) = \sum_{n=1}^{\infty} f(t)\sin(2n-1)x = \sum_{n=1}^{\infty} \frac{Ae^{At}}{Be^{At}+CA}\sin(2n-1)x \tag{19}$$

Here first two boundary conditions of (2) are already verified and third boundary condition will be applied accordingly.

Example: 1 The non-linear parabolic equation (1) and boundary conditions represented by equation (2) with u_0 (x) = x. Taking t = 0 in equation (19), it is obtained as

$$u_0(x) = \sum_{n=1}^{\infty} \frac{A}{B+CA}\sin(2n-1)x \tag{20}$$

By sine series, we have

$$\frac{A}{B+CA} = \frac{2}{\pi}\int_0^{\pi} x \sin(2n-1)x\, dx \tag{21}$$

On integrating above, we have

$$C = \frac{2n-1}{2} - \frac{B}{A} \tag{22}$$

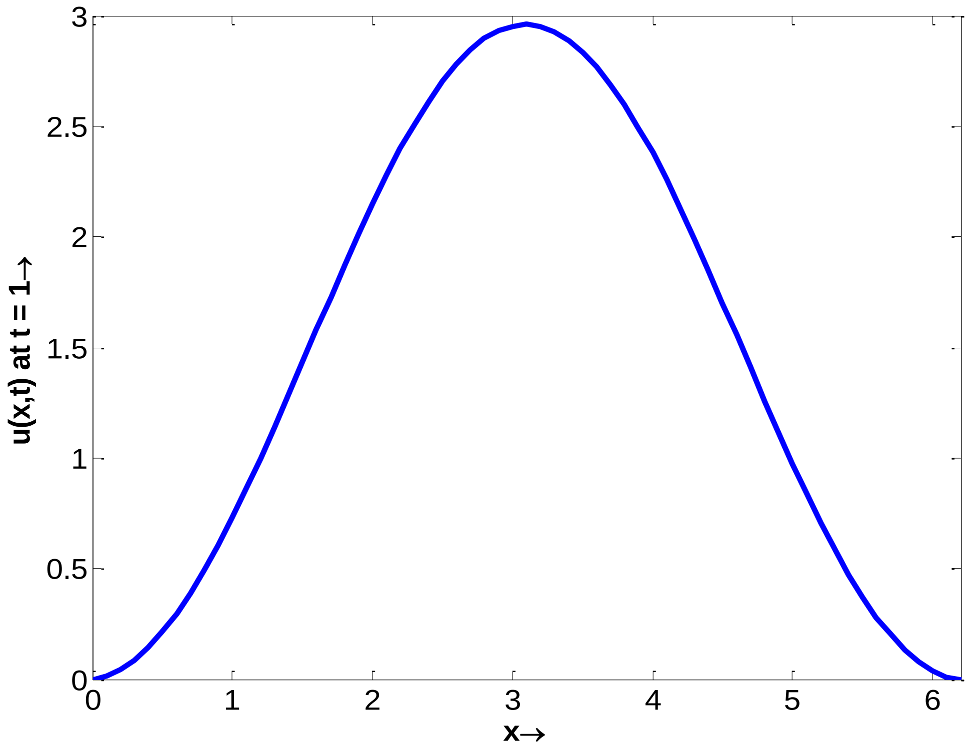

Figure 1 Variation of u(x, t) along the dimension x at t = 1 in Example 1

From equations (18), (19) and (22), it can be expressed aas

$$f(t)=\frac{2Ae^{At}}{B\left(e^{At}-1\right)+(2n-1)A} \tag{23}$$

From equations (19) and (23), we have

$$u(x,t)=\sum_{n=1}^{\infty}\frac{2Ae^{At}}{B\left(e^{At}-1\right)+(2n-1)A}\sin(2n-1)x \tag{24}$$

Example: 2 The non-linear parabolic equation (1) and boundary conditions represented by (2) with $u_0(x) = x^3$. Taking t = 0 in equation (19)

$$u_0(x)=\sum_{n=1}^{\infty}\frac{A}{B+CA}\sin(2n-1)x$$

By sine series

$$\frac{A}{B+CA}=\frac{2}{\pi}\int_0^{\pi}x^3\sin(2n-1)x\,dx \;=\frac{2}{(2n-1)^3}\left(\pi^2(2n-1)^2-6\right)$$

which gives

$$C=\frac{(2n-1)^3}{2}\frac{1}{\pi^2(2n-1)^2-6}-\frac{B}{A} \tag{25}$$

$$u(x,t)=\sum_{n=1}^{\infty}\frac{Ae^{At}}{B\left(e^{At}-1\right)+\frac{(2n-1)^3}{2}\frac{A}{\pi^2(2n-1)^2-6}}\sin(2n-1)x \quad (26)$$

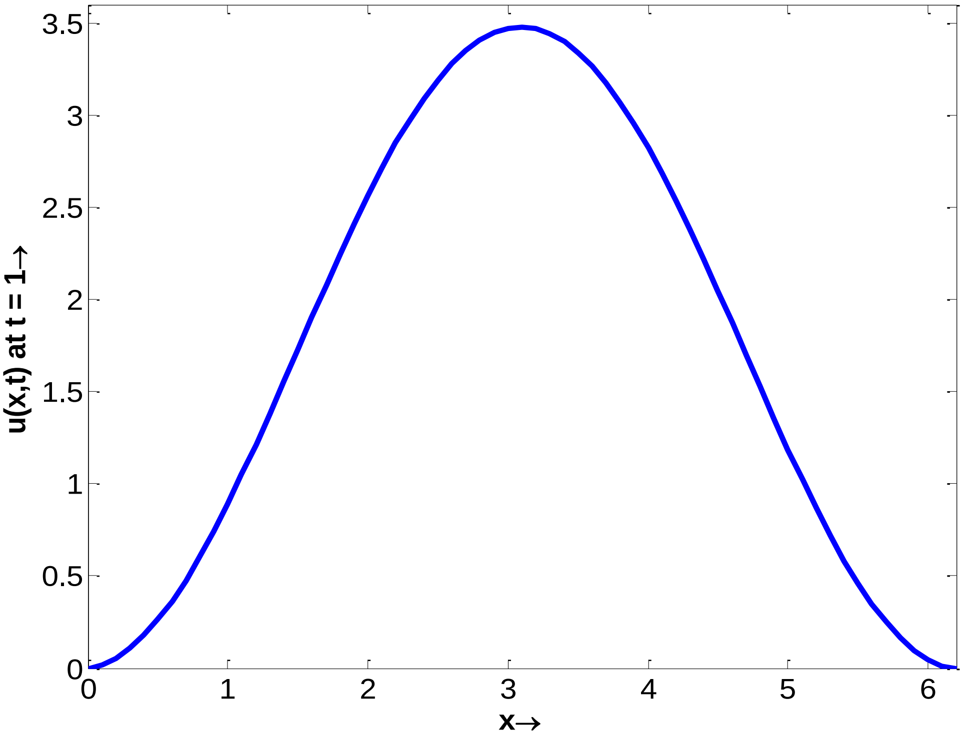

Figure 2 Variation of u(x, t) along the dimension x at t = 1 in Example 2

Example: 3 The non-linear parabolic equation (1) and boundary conditions represented by (2) with $u_0(x) = x \cos x$. Taking t = 0 in (19)

$$u_0(x)=\sum_{n=1}^{\infty}\frac{A}{B+CA}\sin(2n-1)x$$

By sine series

$$\frac{A}{B+CA}=\frac{2}{\pi}\int_0^{\pi}x\cos x\sin(2n-1)x\,dx$$

$$=-\frac{2n-1}{2n(n-1)}+\frac{1}{\pi}\left(\frac{\sin 2nx}{4n^2}+\frac{\sin 2(n-1)x}{4(n-1)^2}\right)_0^{\pi}$$

$$\frac{A}{B+CA}=-\frac{2n-1}{2n(n-1)}\ \text{ for n = 2, 3, 4, ...}$$

By which, we have

$$C = -\left(\frac{2n(n-1)}{2n-1} + \frac{B}{A}\right) \text{ for n = 2, 3, 4, ...} \tag{27}$$

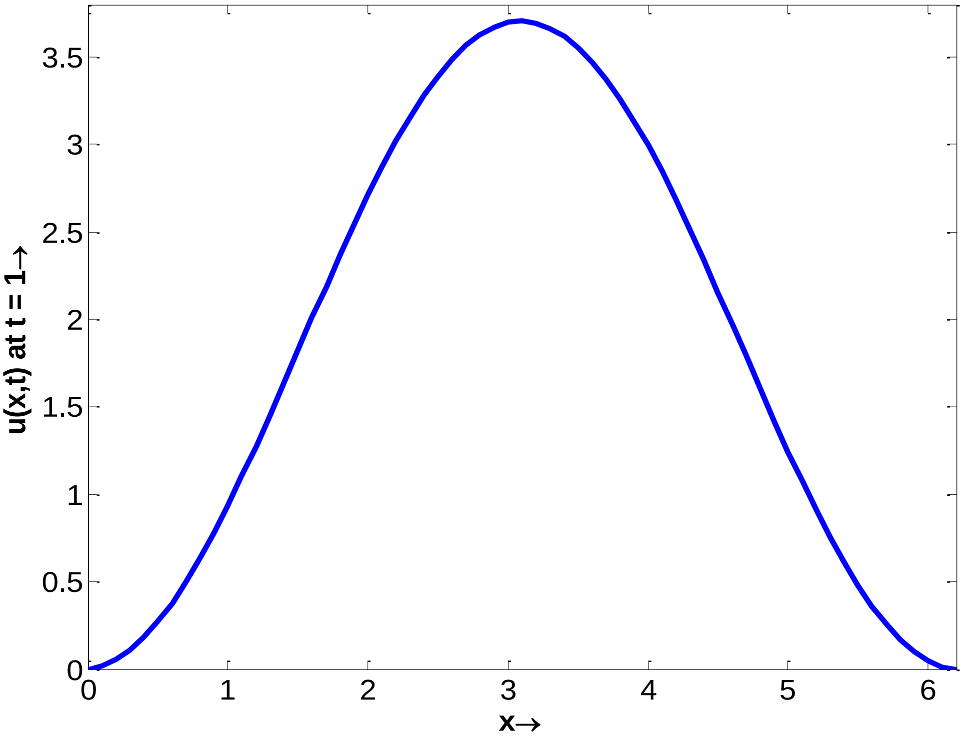

Figure 3 Variation of u(x, t) along the dimension x at t = 1 in Example 3

For n = 1

$$\frac{A}{B+CA} = \frac{1}{\pi}\int_0^\pi x\left(2\sin x \cos x\right)dx = \frac{1}{\pi}\int_0^\pi x\sin 2x \, dx$$

$$= \frac{1}{\pi}\left(-x\frac{\cos 2x}{2} + \frac{\sin 2x}{4}\right)_0^\pi = -\frac{1}{2}$$

By which, we have

$$C = -2 - \frac{B}{A} \tag{28}$$

Then we obtain

$$u\left(x,t\right) = \frac{Ae^{AT}}{B\left(e^{At}-1\right)-2A}\sin x + \sum_{n=2}^{\infty}\frac{Ae^{At}}{B\left(e^{At}-1\right)-\frac{2n\left(n-1\right)}{2n-1}A}\sin\left(2n-1\right)x \tag{29}$$

CONCLUSION

In this paper, the solution of reaction diffusion equation with ecological parameters is obtained by use of holder inequality and integral transform method. The area in which it can be applied is not limited to mathematics only, but it can be widely used by researchers from the field of biology, geology, physics and ecology. It can be used in eco environment system like treating water impurities and purify the air in the atmosphere. In our work, we have used the MATLAB to calculate the series obtain from the integral transform method and to get graphical solution of reaction diffusion equation.

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