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Sums Formulas of Generalized Fibonacci Polynomial Sequence

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Abstract
Here, we work on some finite series of generalized Fibonacci polynomials and its first order derivatives. In the next section, extreme values have been calculated for the Fibonacci polynomials. In the last section, we represent the Fibonacci polynomials in two variables by graphs using MATLAB.

Keyword: **generalized Fibonacci polynomials, extreme values, MATLAB.**

I. INTRODUCTION

The generalized Fibonacci polynomials defined as follows:

$$Q_{n+2}(x) = rxQ_{n+1}(x) + sQ_n(x), \qquad Q_0(x) = a, Q_1(x) = b, \tag{1}$$

where $Q_0(x)$, $Q_1(x)$, a, b, r and s are any real numbers.

If we set $Q_0(x) = 0, Q_1(x) = 1, r = 1$ and $s = 1$, then $\{Q_n(x)\}$ becomes the sequences of Fibonacci polynomials and if we set $Q_0(x) = 2, Q_1(x) = 1$, $r = s = 1$, then it becomes the sequence of Lucas polynomials.

[1] presented the closed forms of the summation formulas for generalized tribonacci numbers. Similar work has been done for different sequences of Fibonacci numbers by [2, 3,4]. Here, we derive some summation formulas of generalized Fibonacci polynomials and its first order derivative.

This paper has mainly three sections. In the first section,we represent the some summation formulas for generalized Fibonacci polynomials and its first order derivative.

. In the second section, work has been done for the extreme values of Fibonacci polynomials. And in the last section, we represent these polynomials in form of graphs using MATLAB.

II. SUM FORMULAS OF GENERALIZED FIBONACCI POLYNOMIALSWITH POSITIVE SUBSCRIPTS

The following theorem presents some summing formulas of generalized Fibonacci polynomials and its first order derivative with positive subscripts.

Theorem 2.1For $n \geq 0$, we have the following formulas for Generalized Fibonacci polynomials:

(a) If $s + rx - 1 \neq 0$, then

$$\sum_{k=0}^{n} Q_k(x) = \frac{\mu}{s + rx - 1}$$

where

$$\mu = Q_{n+2}(x) + (1 - rx)Q_{n+1}(x) + Q_1(x) + (rx - 1)Q_0(x)$$

(b) If $s^2 - r^2x^2 - 2s + 1 \neq 0$, then

$$\sum_{k=0}^{n} Q_{2k}(x) = \frac{\mu}{s^2 - r^2x^2 - 2s + 1}$$

where

$$\mu = (s-1)Q_{2n+2}(x) - rxsQ_{2n+1}(x) + rxQ_1(x) - (r^2x^2 + s - 1)Q_0(x)$$

(c) If $s^2 - r^2x^2 - 2s + 1 \neq 0$, then

$$\sum_{k=0}^{n} Q_{2k+1}(x) = \frac{-\mu}{s^2 - r^2x^2 - 2s + 1 \neq 0,}$$

where

$$\mu = rxQ_{2n+2}(x) - s(s-1)Q_{2n+1}(x) + (s-1)Q_1(x) - rxQ_0(x)$$

Proof of (a)

Using the recurrence relation

$$Q_{n+2}(x) = rxQ_{n+1}(x) + sQ_n(x)$$

i.e.,

$$sQ_n(x) = Q_{n+2}(x) - rxQ_{n+1}(x)$$

We have

$$sQ_0(x) = Q_2(x) - rxQ_1(x)$$
$$sQ_1(x) = Q_3(x) - rxQ_2(x)$$
$$sQ_2(x) = Q_4(x) - rxQ_3(x)$$
$$sQ_3(x) = Q_5(x) - rxQ_4(x)$$
$$\vdots$$
$$sQ_{n-3}(x) = Q_{n-1}(x) - rxQ_{n-2}(x)$$
$$sQ_{n-2}(x) = Q_n(x) - rxQ_{n-1}(x)$$
$$sQ_{n-1}(x) = Q_{n+1}(x) - rxQ_n(x)$$
$$sQ_n(x) = Q_{n+2}(x) - rxQ_{n+1}(x)$$

By adding these equations sides by sides, then we obtain (a).

Proof of (b) and (c)

Using the recurrence relation

$$Q_{n+2}(x) = rxQ_{n+1}(x) + sQ_n(x)$$

i.e.,

$$rxQ_{n+1}(x) = Q_{n+2}(x) - sQ_n(x)$$

We have

$$rxQ_1(x) = Q_2(x) - sQ_0(x)$$

$$rxQ_3(x) = Q_4(x) - sQ_2(x)$$
$$rxQ_5(x) = Q_6(x) - sQ_4(x)$$
$$rxQ_7(x) = Q_8(x) - sQ_6(x)$$
$$\vdots$$
$$rxQ_{2n-1}(x) = Q_{2n}(x) - sQ_{2n-2}(x)$$
$$rxQ_{2n+1}(x) = Q_{2n+2}(x) - sQ_{2n}(x)$$

Now, by adding these side by side, we have

$$rx\sum_{k=0}^{n} Q_{2k+1}(x) = Q_{2n+2}(x) + \left(\sum_{k=0}^{n} Q_{2k}(x) - Q_0(x)\right) - s\left(\sum_{k=0}^{n} Q_{2k}(x)\right) \tag{2}$$

Again using the recurrence relation

$$Q_{n+2}(x) = rxQ_{n+1}(x) + sQ_n(x)$$

i.e.,

$$sQ_n(x) = Q_{n+2}(x) - rxQ_{n+1}(x)$$

We write the following obvious equations;

$$rxQ_2(x) = Q_3(x) - sQ_1(x)$$
$$rxQ_4(x) = Q_5(x) - sQ_3(x)$$
$$rxQ_4(x) = Q_7(x) - sQ_5(x)$$
$$\vdots$$
$$rxQ_{2n-2}(x) = Q_{2n-1}(x) - sQ_{2n-3}(x)$$
$$rxQ_{2n}(x) = Q_{2n+1}(x) - sQ_{2n-1}(x)$$

Now, by adding these side by side, we have

$$rx(-Q_0(x) + \textstyle\sum_{k=0}^{n} Q_{2k}(x)) = \left(\sum_{k=0}^{n} Q_{2k+1}(x) - Q_1(x)\right) - s\left(\sum_{k=0}^{n} Q_{2k+1}(x) - Q_{2n+1}(x)\right) \tag{3}$$

On solving the equations (2)and (3),the proof of (b) and (c) follows.

Theorem 2.2 For $n \geq 0$, the following formulas hold for 1st order derivatives of generalized Fibonacci polynomials:

(a) If $(s + rx - 1)^2 \neq 0$, then

$$\sum_{k=0}^{n} Q_k{}'(x) = \frac{\mu}{(s + rx - 1)^2}$$

where

$$\mu = (s + rx - 1)Q_{n+2}{}'(x) - rQ_{n+2}(x) - (rx - 1)(s + rx - 1)Q_{n+1}{}'(x)$$
$$-r(1 - rx)Q_{n+1}(x) - (s + rx - 1)Q_1{}'(x) - rQ_1(x) + (rx - 1)(s + rx - 1)Q_0{}'(x) + rsQ_0(x)$$

(b) If$(s^2 - r^2x^2 - 2s + 1) \neq 0$, then

$$\sum_{k=0}^{n} Q_{2k}'(x) = \frac{\mu}{(s^2 - r^2x^2 - 2s + 1)^2}$$

where

$$\mu = \rho(s-1)Q_{2n+2}'(x) + r(2xrs - 2xr - s^3 - s + 2s^2 + r^2x^2s)Q_{2n+1}(x) - rx\rho Q_{2n+1}'(x) + r(s^2 + 1 - 2s + r^2x^2)Q_1(x) + rx\rho Q_1'(x) - 2xr^2s(s-1)Q_0(x) - \rho(r^2x^2 + s - 1)Q_0'(x)$$

and $\rho = s^2 - r^2x^2 - 2s + 1$

(c) If $s^2 - r^2x^2 - 2s + 1 \neq 0$, then

$$\sum_{k=0}^{n} Q_{2k+1}'(x) = \frac{\mu}{\rho^2}$$

where

$$\mu = \rho rx Q_{2n+2}'(x) - r(r^2x^2 + (s-1)^2)Q_{2n+2}(x) + 2xsr^2(s-1)Q_0 Q_{2n+1}(x) - s(s-1)\rho Q_{2n+1}'(x) - 2r^2x(s-1)Q_1(x) + (s-1)\rho Q_1'(x) + r(r^2x^2 + (s-1)^2)Q_0(x) - \rho rx Q_0'(x)$$

where

$$\rho = -(s^2 - r^2x^2 - 2s + 1)$$

Proof: Differentiating the above results of Theorem 2.1 with respect to x, these can be easily proved.

Corollary 2.1 For $n \geq 0$, the following formulas holds for Fibonacci sequence of numbers:

(a) $\sum_{k=0}^{n} Q_k = Q_{n+2} + Q_1$

(b) $\sum_{k=0}^{n} Q_{2k} = Q_{2n+1} - Q_1 + Q_0$

(c) $\sum_{k=0}^{n} Q_{2k+1} = Q_{n+2} - Q_0$

Corollary 2.2 For $r = s = 1$ $n \geq 0$, the following formulas holds for Tetrabonacci sequence:

(a) $\sum_{k=0}^{n} Q_k'(1) = Q_{n+2}'(1) - Q_{n+2}(1) - Q_1'(1) - Q_1(1) + Q_0(1)$

(b) $\sum_{k=0}^{n} Q_{2k}'(1) = Q_{2n+1}'(1) + Q_{2n+1}(1) - Q_1'(1) + Q_1(1) + Q_0'(1)$

(c) $\sum_{k=0}^{n} Q_{2k+1}'(1) = -Q_{2n+1}(1) - Q_0'(1) + Q_0(1)$

Corollary 2.3 For $r = 2, s = 1,$ *and* $n \geq 0$, the following formulas holds for Pell numbers:

(a) $\sum_{k=0}^{n} Q_k = \frac{Q_{n+2} - Q_{n+1} + Q_1 + Q_0}{2}$

(b) $\sum_{k=0}^{n} Q_{2k} = \frac{Q_{2n+1} - Q_1 + 2Q_0}{2}$

(c) $\sum_{k=0}^{n} Q_{2k} = \frac{Q_{n+2} - Q_0}{2}$

Next, we will express the solution of an ordinary differential equation in terms of Fibonacci numbers and also determine the extreme values of the Fibonacci polynomials of one variable and two variables.

The Fibonacci polynomials $F_n(x)$ are defined by

$F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$for $n \geq 2$(4)

with $F_1(x) = 1$, $F_2(x) = x$.

The Fibonacci polynomials $F_n(x)$ satisfy the second order linear [5] differential equation

$$(x^2 + 4)y'' + 3xy' - (n^2 - 1)y = 0 \quad (5)$$

Also, the Fibonacci polynomials in two variables [6] are defined by the following

$F_n(z',t') = z'F_{n-1}(z',t') + t'F_{n-2}(z',t')$for $n \geq 2$(6)
with $F_1(z',t') = 1$, $F_2(z',t') = z'$.

III. Solution of a Differential equation in terms of Fibonacci numbers

Consider the following second order linear homogenous differential equation

$$y'' - y' - y = 0 \quad (7)$$

On solving this, its solution is given by

$$y = c_1 e^{\alpha x} + c_2 e^{\beta x} \quad (8)$$

where c_1 and c_2 are arbitrary constants and $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

If we take $c_1 = 1$ and $c_2 = -1$, then solution (8) becomes

$$y = e^{\alpha x} - e^{\beta x} \quad (9)$$

Now, using the series expansion of exponential function, (9) becomes

$$y = (\alpha - \beta)x + \frac{(\alpha^2 - \beta^2)x^2}{2!} + \frac{(\alpha^3 - \beta^3)x^3}{3!} + \frac{(\alpha^4 - \beta^4)x^4}{4!} + \cdots \qquad (10)$$

By using the Binet formula of Fibonacci polynomials, (10) becomes

$$y = x + \frac{(\alpha^2 - \beta^2)x^2}{2!\,(\alpha - \beta)} + \frac{(\alpha^3 - \beta^3)x^3}{3!\,(\alpha - \beta)} + \frac{(\alpha^4 - \beta^4)x^4}{4!\,(\alpha - \beta)} + \cdots$$

$$= F_0 + F_1 x + \frac{F_2 x^2}{2!} + \frac{F_3 x^3}{3!} + \frac{F_4 x^4}{4!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{F_n x^n}{n!}$$

Now, the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{F_n x^n}{n!}$ is infinite, therefore the solution of the differential equation (7) converges for all real values of x.

IV. Extreme Values

From the recurrence relation(4), we observe that for odd values of n, Fibonacci polynomial sequence has no real roots by Descart's rule of sign. And for even n, it has only one real root equal to zero. Again by the Descart's rule of sign, we observe that there is only one critical point equal to zero for the Fibonacci polynomial sequence having odd indices and there is no critical point for polynomials sequence having even indices(4), therefore have no extreme values. On the other hand, Fibonacci polynomials having odd indices has positive second derivative at $x = 0$, therefore having minima at $x = 0$.Thus , we can conclude that Fibonacci polynomials defined by (4) has

1. no extreme value for even n
2. Only one extreme value for odd n.

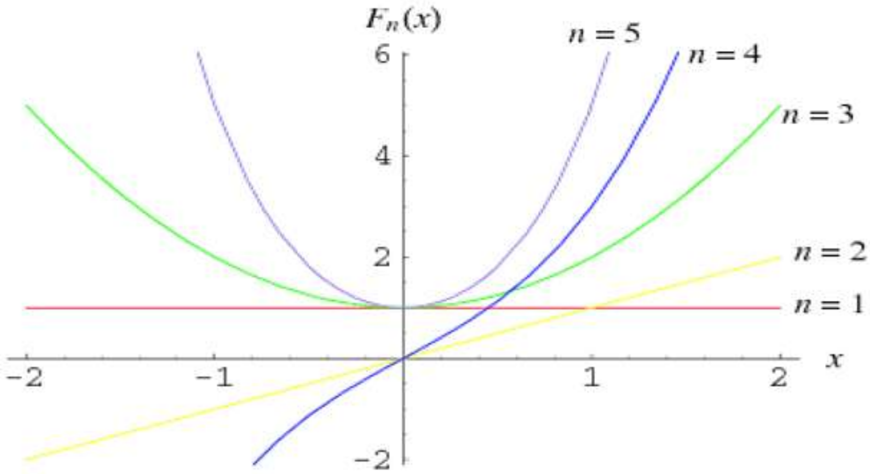

Figure 4.1: Graph for Fibonacci polynomial

Next, we will find out the extreme value for the Fibonacci polynomials in two variables defined by (6) for $n = 1$ to 7.Now, the necessary condition for a function $f(x, y)$ of two variables to have an extremum at (a, b) are that

$\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

We observe that there is no critical point for $n = 1,2,3$. And for $n = 4,5,6,7$, there is only one critical point (0,0).

Now, for $n = 4$

$F_4(z', t') - F_4(0,0) < 0$ for $z' = -0.0001$ and $t' = 0.001$

And $F_4(z', t') - F_4(0,0) > 0$ for $z' = 0.0001$ and $t' = 0.001$, therefore $F_4(z', t')$ has neither maxima nor minima at (0,0).

Now, for $n = 5$

$F_5(z', t') - F_5(0,0) < 0$ for $z' = 0.01$ and $t' = -0.001$

And $F_5(z', t') - F_5(0,0) > 0$ for $z' = 0.01$ and $t' = 0.001$, therefore $F_5(z', t')$ has neither maxima nor minima at (0,0).

Now, for $n = 6$

$F_6(z', t') - F_6(0,0) < 0$ for$z' = -0.01$ and $t' = 0.001$

And $F_6(z', t') - F_6(0,0) > 0$ for $z' = 0.01$ and $t' = 0.001$, therefore $F_6(z', t')$ has neither maxima nor minima at (0,0).

Now, for $n = 7$

$F_7(z', t') - F_7(0,0) < 0$ for$z' = 0.001$ and $t' = -0.00001$

And $F_7(z', t') - F_7(0,0) > 0$ for $x = 0.01$ and $y = 0.001$, therefore $F_7(z', t')$ has neither maxima nor minima at (0,0). Thus, Fibonacci polynomial defined by (6) has no extreme values for $n = 1$ to 7

V. GRAPHS USING MATLAB

Now, we will plot the graph of Fibonacci polynomials in two variables defined by (6) , for $n = 3, 4, \dots, 7$ using MATLAB.

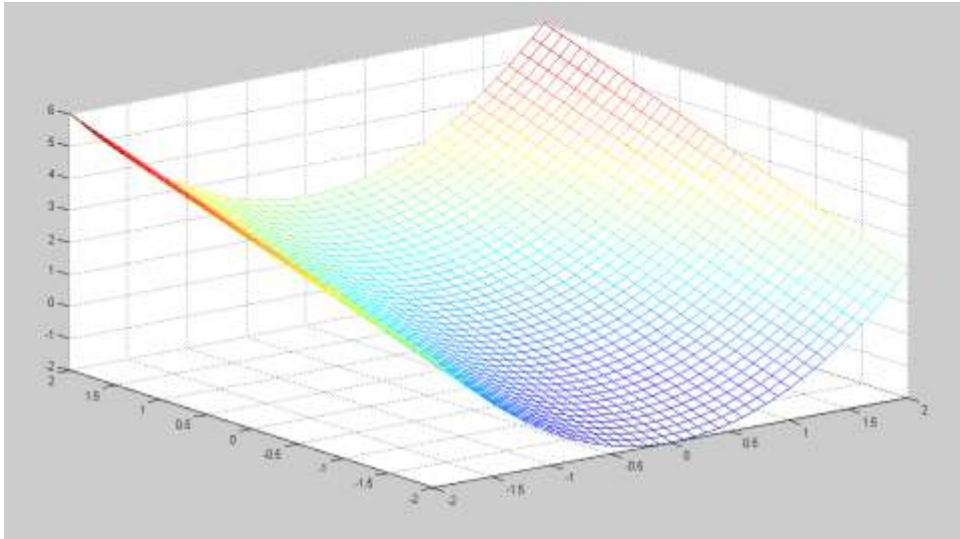

Figure 5.1 Graph for $F_3(z', t') = z'^2 + t'$

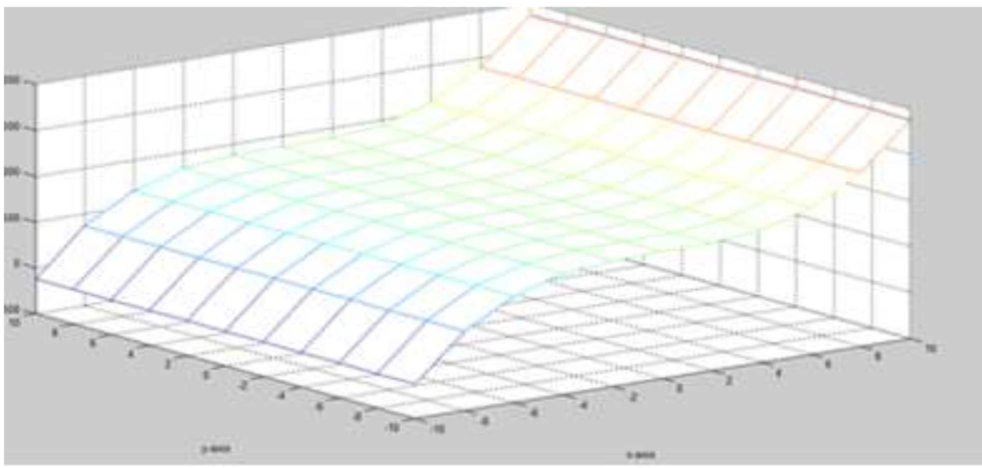

Figure 5.2 Graph for $F_4(z', t') = z'^3 + 2z't'$

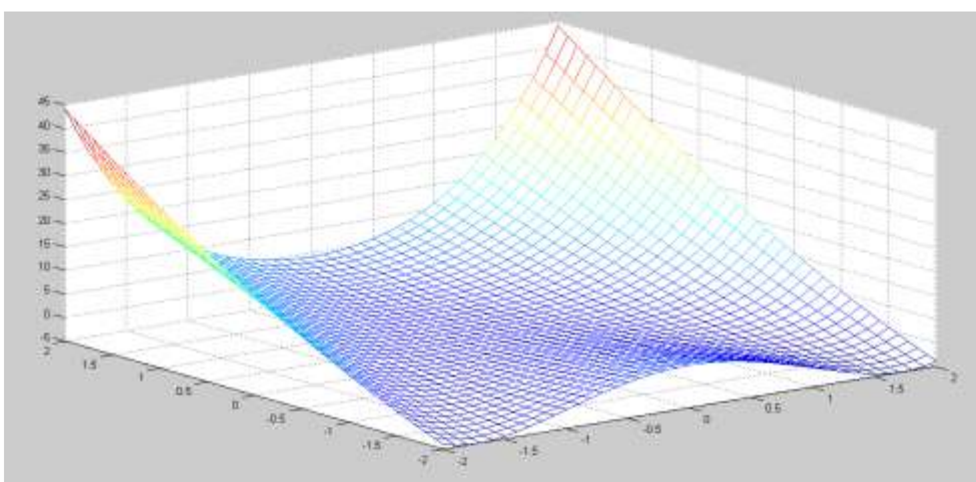

Figure 5.3 Graph for $F_5(z', t') = z'^4 + 3z'^2t' + t'^2$

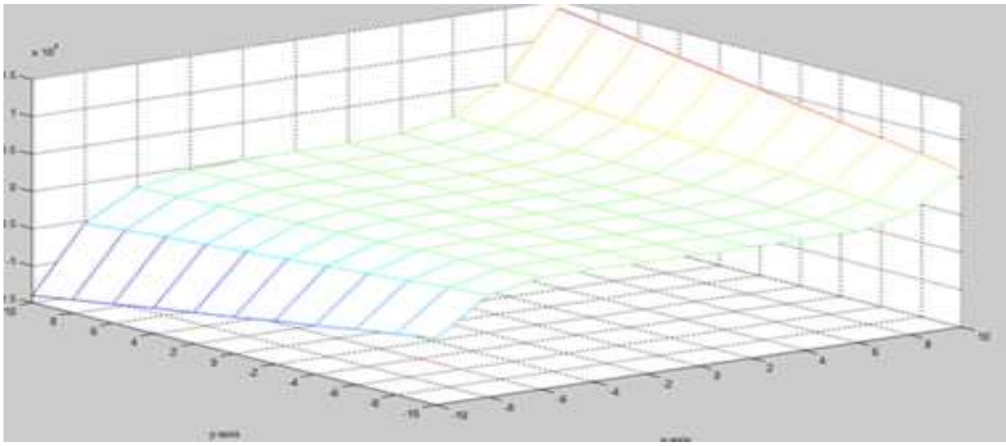

Figure 5.4 Graph for $F_6(z',t') = z'^5 + 4z'^3t' + 3z't'^2$

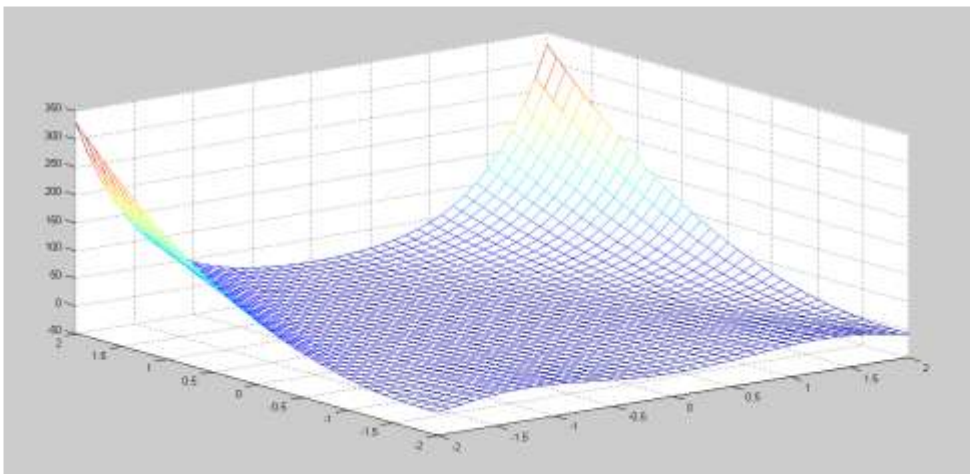

Figure 5.5 Graph for $F_6(z',t') = z'^6 + 5z'^3t' + 6z'^2t'^2 + t'^3$

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