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On Studying Some Modular Modulatin Spaces

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ABSTRACT- In the present paper we propose a new algebraic structure on some famous Banach spaces called modulation space. Our suggestion goes through an analytical properties of a Banach space which would be equipped with a weight function and then a convolution. Furthure we consider the operator theory scheme of these spaces and discus more valuable and applicable properties which will be arise in quantization formulations. The subject here would consider as some elementary properties for any modulation space.

Keywords: Modular Properties, Modulation Spaces, Gabor frames

I. INTRODUCTION

The modulation spaces, are well known now. They were invented by H. G. Feichtinger and as a concrete frame work for time-frequency analysis, they have found their place in several branches of science and engineering and any where a function would be analysis in its different propertie (Bastos et al., 2009)and (Boggiatto et al., 2019). These classes of function spaces which are based on measuring both time and frequency concentration of a function(Dias et al., 2016) and(Feichtinger et al., 1992), are one of the more interesting areas of modern analysis attract many researchers to consider(Grochenig & Heil, 1999). As a useful and also almost simple tool, modulation spaces are a fast and exact framework to work for mathematicians and also for engineers for many kind of analytical and engineering process such as signal processing or image processing and other applications that their problems may come to harmonic analysis like video image processing, quantization forms, operator theory field, communications, frame theory , etc.,. One can finds easily in the wide area of papers and books, for example (Bastos et al., 2008) and (Brandenburg, 1975).
An important tools for employing modulation spaces as the mentioned frame work, are Wilson bases, their synthesis properties in weighed case, Gabor frames and operator representation of modulation spaces that many researchers have used them for investigating several results in different areas(Feichtinger, 2006) and (Feichtinger & Kozek, 1998). Our work also is based on these tools to imply a new product on a modulation space as a Banach spaces which we will use it then, like an algebraic Banach space to characterize some properties of these spaces. Also we will describe and answer the Feightinger M^1 question (Cappiello & Toft, 2017).

Notations and Preliminaries
Through this work H is an infinite-dimensional separable Hilbert space. Then · denotes its norm and ·
is its inner product. For vectors x and y in H, (x y) is rank one operator
(x y)z= z, $y$$x$ which its operator norm is $x$$y$. We denote by $x \cdot y$,the scalar product and abbreviate x^{2}= $x \cdot x$ by x^2. The Schwartz class is denoted by S and its dual, the space of tempered distributions, by S'. If A $B(H)$ is a compact operator then we show its eigenvalues with λ_j and its singular values with $s_j(A)$. Sequence $\{s_k(A)\}_{k=1}\infty$ is non increasing. For a Banach space B and a weight function m we denote by B_m the weighted Banach space $\{ f$$B$: $f m$ $B\}$ with norm f B_m= $f m$$_B$ for f B.

II. METHOD

Our method here is based on a constructive structure with a moderate weight function and then defining a product on our modulation space which is made from time-frequency shift functions.

1. WEIGHTS AND MODULATION SPACES

Generally a *weight* v or m is a non-negative continuous function on R^{2d}. Herewe mention some famous class of weight functions :
(i) v is called *submultiplicative*, if $v(z + z') \leq v(z)v(z')$ for all z, z' R^{2d};
(ii) m is called *v-moderate* with respect to a weight v, if $m(z + z') \leq Cv(z)m(z')$ for some positive constant C and all z, z' R^{2d};
$v_s(z) = (1+|z|^2)^{s/2} = (1+x^2 +w^2)^{s/2}$, $z = (x,w)$ R^{2d}. (1.1)

The weights v_s appear naturally in different branches of mathematics and physics, e.g. Banach convolution algebras, function spaces, harmonic analysis, pseudodifferential operators, classes of matrix algebras and quantum mechanics. The weights v_s have many properties that make them of interest. We recall some of these properties in the following lemma.

Lemma 1.1. *For every polynomial weight vs we have*
(i) *vs is submultiplicative for all* $s \geq 0$*;*
(ii) *If* $0 \leq t \leq s$*, then vs and* $v-1$
t are v-moderate;

Fundamental object in modulation spaces structure is *short time Fourier transform*(STFT) which arises from unitary operators T_x(translation) and M_w(modulation)onL^2(R):
$T_x f(t) = f(t-x)$ and $M_w f(t) = e^{(2\pi it)} f(t)$, (1.2)

as time and frequency shifts. *Short time Fourier transform* (STFT) of function *f*with respect to function *g*, which is called window function or briefly window, isdefined to be $V_g f(x,w)$ such that$V_g f(x,w) = ⟨f, M_w T_x g⟩$, with classical L^2 inner product,whenever the integral or the inner product exists, e.g. for (f, g) $\in L^2(R^d) \times L^2(R^d)$or $(f, g) \in S(R^d) \times S'(R^d)$.

Definition 1.2. For a non-zero window $g \in S(R^d)$, a *v*-moderate weight function *m* on $R^{d2}$and $1 \leq p < \infty$, the modulation space $M^p_m(R^d)$ consists of all tempered distributions $f \in S'(R^d)$ such that$V_g f \in L^p_m(R^{d2})$. The norm on $M^p_m(R^d)$is$\|f\|_{Mpm} = \|Vg f\|_{Lpm}$ *with* obvious changes if $p = \infty$.
In order to find the basic properties of this family of function spaces we need tounderstand the inversion formula for the STFT. Consider the operator
$f = \frac{1}{\langle \gamma, g \rangle} \iint_{R^{2d}} V_g f(x,w) M_w T_x \gamma \, dwdx,$
Which describe the inversion formula for *f,* see (Dias et al., 2010) and(De Gosson & Luef,2010).

Historically, extending the theory of Gabor frames from $L^2$to modulation spaces M^p, provided a very strong and flexible framework through modulation spaces.

If $G(g,a,b)$ is a frame for $L^2(R^d)$, it is called a Gabor frame. Following important fact leads to a characterization of modulation spaces by means of Gabor frames.
Proposition 1.3. *Let v be a moderate weight on* R^d *. If* $G(g,a,b)$ *is a Gaborframe for* $L^2(R^d)$*with* $g \in M^1_v(R^d)$*, then* $G(g,a,b)$ *is a Banach frame for* $M^p(R^d)$*,*$1 \leq p \leq \infty$*. Consequently, each f in* $M^p_m(R^d)$*has a discrete representation with respect to a dual window* $\in M^1_v(R^d)$*, i.e.,*
$f = \Sigma\Sigma ⟨f, T_k M_n⟩ T_k M_n g$, for *k,n in Z,*
that converges unconditional for some$1 \leq p < \infty$ *and with weak*-converge in* $M\infty = v(R^d)$.
for all $f \in M^p_m(R^d)$ *with v-moderate weight m.*
Now, consider Gabor system $G(g, 2Z, Z)$ be a Gabor system of redundancy 2 in
$L^2(R^d)$. Then the associated *Wilson system*, $W(g)$, consists of the symmetric linear combinations of time and frequency shifts of function *g*:
$k_n = c_n T_k = 2(M_n + (-1)^{(k+n)} M_{-n})g$ for $k, n \in Z$ and $n \geq 0$.

Theorem 1.4. Let g be in $L^2(R^d)$ such that $g = g*$ and $\|g\|2 = 1$. If $G(g, 2Z, Z)$ is a tight Gabor frame for $L^2(R^d)$, thenW(g) is an orthonormalWilson basis of $L^2(R^d)$. (Daubechies et al., 1991).
Theorem 1.5. Assume that W(g) is an orthonormal basis for $L^2(R)$ with $g \in M^1 v(R)$ for a moderate weight function *v* of polynomial weights. Then

$$\frac{1}{C}\|f\|_{M^p_m} \leq \left(\sum_{n=0}^{\infty}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{kn}\rangle|^p \, m(\tfrac{k}{2}, n)^p\right)^{1/p} \leq C\|f\|_{M^p_m}.$$

for a constant $C \geq 1$:

$$f = \sum_{n=0}^{\infty}\sum_{k\in\mathbb{Z}} \langle f, \psi_{kn}\rangle \psi_{kn},$$

The proof could be find in (Grochenig, 1996) and (Nikolski, 1970).
Theorem 1.6. Let m be a *v*-moderate weight and $W(g) = \{\psi_{k,n}: k, n \in Z, n \geq 0\}$.

Then { $\psi_{rs}= \Pi^d_{i=1}\, \psi_{ri,si}(x_i)$} is an unconditional basis for $M^p{}_m(R^d)$for 1 ≤ p <∞ and weak-∗-convergent in $M^1v(R^d)$ otherwise.

2.INTEGRAL OPERATORS WITH SYMBOLS IN WEIGHTED MODULATION SPACES

Wilson basis is a usefultool in various proofs of basic properties of modulation spaces, e.g. in theisomorphism theorem for modulation spaces. The kernel theorem says that for a distributional symbol k∈$M^1v(R^d)$, the operator $A_k$is a map from$M^1v(R^d)$ to $M\infty v(R^d)$.

Theorem 2.1. Every distribution k ∈ $M\infty v(R^d)$ defines a bounded operator
A_k:$M^1v(R^d)\rightarrow M\infty v(R^d)$by
$\langle A_k f , g\rangle= \langle k, g \otimes f \rangle$, f , g ∈$M^1v(R^d)$,
and vise-versa.

So investigation properties of integral operator $A_k$could be studied from itskernel. Thanks to work of Labate, we can go from Schatten norm of $_kA$toMp norm of k, (Heil, 2013).

Definition 2.2. Let B be a Banach space, T ∈$L(B)$, and $1 \le p <\infty$. Then T
is absolutely p-summing if there is a constant $c \ge 0$ such that for all sequences
$\{(f_i)_{i=1}{}^m\}$in B:

$$\left(\sum \|Tf_i\|_B^p\right)^{1/p} \le c \sup_{g\in L(B*)} \left(\sum |\langle g, f_i\rangle|^p\right)^{1/p}.$$

Let $p(T)$ = inf$c$, for which above inequality holds. Then the collection $p(X)$, of p-summing operators on B is a Banach space with norm$p(T)$.

Remark 2.3. If B is the Hilbert space H, then $p(H)$ coincides with the class I_2 of Hilbert-Schmidt operators on H for p ∈[1,∞], (Grochenig, 2001) and(Grochenig, 2018),

Proposition 2.4. Let 1 ≤ p <∞ and p′ be the conjugate exponent of p.

(i) If 1 ≤ p ≤ 2 and k ∈$M^p(R^d)$ then A_k is a compact operator (for p = 1), A_k is weakly compact and completely continuous operator which maps $M^p(R^d)$ into itself. Singular values of A_k are 2-summable and

$$\|A_k\|_{\mathscr{I}_2} = \left(\sum s_j(A_k)^2\right)^{1/2} \le C\|k\|_{M^p},$$

(ii) If 2 ≤ p <∞ and k ∈$M^{p'}(R^d)$ then A_k is a weakly compact and completely continuous operator which maps $M^{p'}(R^d)$ into itself. Singular values of A_k are r-summable such that r = max{2, p} and

$$\|A_k\|_{\mathscr{I}_r} = \left(\sum s_j(A_k)^r\right)^{1/r} \le C\|k\|_{M^{p'}},$$

Theorem 2.5. For integral kernel k ∈$M^p{}_m(R^d)$and corresponding integral operator
$A_k f (x) =\int k(x, y) f (y)d y$ and s >0, the following statements hold for singular values
$\{s_N(A_k)\}$:
If 1 ≤ p ≤ 2 and k ∈$M^p{}_m(R^d)$, $s_N(A_k)\}= O(N^{-s/2d-1/2})$,(Luef F. & Rahbani, 2011).

Now we can state one of our main results:

Proposition 2.6. If k ∈$M^1(R^{d2})$ then the corresponding integral operator A_k is
trace-class.

Proof. Let W(g) = {ψ_n:n∈N }be aWilson basis for $L^2 (R^d)$ such that g ∈$M^1(R^d)$. So
W(g) is an unconditional basis for $M^1(R^d)$. By tensor product {ψ_{mn}(x, y)={ψ_m(x)ψ_n(y), m,n in Z} is an orthonormal basis for $L^2 (R^{d2})$ and so is an unconditional basis for$M^1(R^d)$.. Therefore

$$k = \sum_{m,n\in\mathbb{Z}} \langle k, \Psi_{mn}\rangle \Psi_{mn},$$

which series converge in M_1-norm and

$$\|k\|_{M_1} = \left|\sum_{m,n\in\mathbb{Z}} \langle k, \Psi_{mn}\rangle\right| < \infty$$

Definition of integral operator implies that

$$A_k f(x) = \int_{\mathbb{R}} \sum_{m,n\in\mathbb{Z}} \langle k, \Psi_{mn}\rangle \Psi_{mn}(x,y) f(y)dy$$

$$= \sum_{m,n\in\mathbb{Z}} \langle k, \Psi_{mn}\rangle \int_{\mathbb{R}} \psi_m(x)\overline{\psi_n(y)} f(y)dy$$

$$= \sum_{m,n\in\mathbb{Z}} \langle k, \Psi_{mn}\rangle \langle f, \psi_n\rangle \psi_m(x)$$

$$= \sum_{m,n\in\mathbb{Z}} \langle k, \Psi_{mn}\rangle (\psi_m \otimes \psi_n)(f)(x)$$

Since series is absolute convergence, interchanging is valid. Therefore

$$A = \sum_{m,n\in\mathbb{Z}} \langle k, \Psi_{mn}\rangle (\psi_m \otimes \psi_n)$$

In the last equation each operator (m⊗ n) is trace-class and the scalars ⟨k, m,n⟩are summable, so A_k is in I_1.

3.FEICHTINGER QUESTION VIA OPERATOR THEORY

Let A_k be a positive integral operator whose kernel k lies inM^1. Considering spectral representation of A_k, must it be true that

$$\sum_{n=1}^{\infty} \|h_j\|^2_{M^1} < \infty?$$

We prove the following result.

Recall that here $H = L^2(R^d) = M^2(R^d)$ and $M^p{}_m$ □L^2 for $1 \le p \le 2$. Also $W(g)$is Wilson orthonormal basis for L^2.

Theorem 3.1. Let A_k be an integral operator which its kernel k lies in $M^p{}_m$, for $1 \le p \le 2$. If h_j are as above then $\{h_j\}$ are in $M^p{}_m$, for some v-moderate weight m .

Proof.

$$A_k = \sum_{m,n\in\mathbb{Z}} \langle k, \Psi_{mn}\rangle (\psi_m \otimes \psi_n).$$

for each eigenvalue $\lambda_j$we have

$$\lambda_j h_j = A_k h_j = \sum_{m,n\in\mathbb{N}} \langle k, \Psi_{mn}\rangle \langle h_j, \psi_m\rangle \psi_n.$$

Since $\|h_j\| = \lambda_j^{1/2} > 0$, $\|\psi_m\| = 1$ and $\|\psi_n\|_{M^p_m} < C$ for some constant C, then

$$|\lambda_j| \|h_j\|_{M^p_m} = \|A_k h_j\|_{M^p_m} \le \sum_{m,n\in\mathbb{N}} |\langle j, \Psi_{mn}\rangle| |\langle h_j, \psi_m\rangle| \|\psi_n\|_{M^p_m}$$

$$\le \sum_{m,n\in\mathbb{N}} |\langle k, \Psi_{mn}\rangle| \|h_j\| \|\psi_m\| \|\psi_n\|_{M^p_m}$$

$$\le C\lambda_j^{1/2} \sum_{m,n\in\mathbb{N}} |\langle k, \Psi_{mn}\rangle| < \infty.$$

Proposition 3.2. Let A_k be a finite rank positive self adjoint operator with kernelk in $M^p{}_s(R^{2d})$. Then

$$\sum_j \|h_j\|^2_{M^p_s} \le \sum_j j^{\frac{s+d}{2d}}$$

Proof. Above argument shows that

$$\sum_j \|h_j\|^2_{M^p_m} \leq C \sum_j \frac{1}{\lambda_j} \leq C \sum_j \frac{1}{s_j(A_k)},$$

Proposition 3.3. *If A_k is a finite rank positive self adjoint operator with kernel k in $M^p{}_s$*(R^{2d}). For 1<p<2, then

$$\sum_j \|h_j\|^2_{M^p_s} < \infty.$$

The proof is a direct result of above theorem.

III. RESULTS

MAIN RESULTS

Based on time frequency analysis and pseduodifferential operators, the Schrodinger representation of the Heisenberg group acts on L^2 (R^d) by means of the unitary operators.

Consequently the composition law of π's is as follows$\pi(z)\pi(z') = e^{2i\pi\Omega(z,z')}\pi(z')\pi(z)$

where $\Omega(z,z')$ denotes the standard symplectic form on R^{2d}.

Many researchers use in the definition of the Schrodinger representation ρ an alternativeuni modular complex numbers instead of $e^{2i\pi}$. For the discussionof Weyl operators our particular choice appears to be most suitable.

Via integration the irreducible unitary Schrodinger representation of the Heisenberggroup gives rise to a class of operators

$$A = \iint_{\mathbb{R}^{2d}} a(z)\rho(z)dz$$

for a *a* in L^1 (R^{d2}). Let a denote the symplectic Fourier transform of *a* in L^1 (R^{d2}), i.e.,

$$a_\Omega(z) = \iint_{\mathbb{R}^{2d}} a(z')\, e^{2\pi i\Omega(z,z')}dz'$$

The operator associated to L_a is

$$L_a = \iint_{\mathbb{R}^{2d}} a_\Omega(z)\rho(z)dz$$

Which is the famous Weyl correspondence and *a* is called the Weyl symbol of the pseudodifferential operator L_a.

The Weyl correspondence associates the following integral operator to a symbol a

$$L_a f(x) = \iint a(\tfrac{x+y}{2}, \omega)e^{2\pi i(x-y)\omega}f(y)dyd\omega$$

Research on quantum mechanical systems modeling charged particles in uniformmagnetic fields, deformation quantization and noncommutative quantum mechanics suggests new quantization rules that yields to a Weyl calculus on phase spaceand de Gosson called it Landau-Weyl correspondence to emphasis the connection to the Landau levels in the quantum mechanics of charged particles.

We briefly review the basic notions underlying Landau-Weyl calculus. To this end we first introduce a Schrodinger representation of R^{2d} onL^2 (R^{d2}),

$$\tilde{\rho}(z')F(z) = e^{2\pi i\Omega(z,z')}F(z - z') \text{ for } F \in L^2(\mathbb{R}^{2d}),$$

and by an elementary calculation we have that

$$\tilde{\rho}(z_1 + z_2) = e^{-2\pi i\Omega(z_1,z_2)}\tilde{\rho}(z_1)\tilde{\rho}(z_2) \text{ for all } z_1, z_2 \in \mathbb{R}^{2d}.$$

The integrated representation of $\tilde{\rho}$ for a in L^1 (R^{d2}), gives a class of operators

$$\tilde{L}_a = \iint_{\mathbb{R}^{2d}} a_\Omega(z)\tilde{\rho}(z)dz$$

that are called Landau-Weyl operators.

The inter twiner between the two unitary representations ρ and $\tilde{\rho}$ of the Heisenberg group is the windowed wave packet transform

$$\mathcal{U}_\varphi g(z) = 2^d W(g,\varphi)(\tfrac{z}{2}) \text{ for } \varphi, g \in L^2(\mathbb{R}^d)$$

An elementary computation establishes that U' is an inter twiner between the two representations. For later reference we collect these observations in the following lemma.

Lemma 4.1. Let a be a Weyl symbol. Then we have

$$\mathcal{U}_\varphi^\star \tilde{L}_a \mathcal{U}_\varphi = L_a$$

Modulation spaces and coorbit theory specialized to the Schrodinger representation of the Heisenberg group provides the following definition of modulation spaces.
Wermer proved that $L^p_s(\mathbb{R}^{2d})$. is a convolution algebra, if it satisfies the Nikolskii-Wermer condition In the case of the $M^p_s(\mathbb{R}^{2d})$. Nikolskii demonstrated the sufficiency of it, as well as the necessity of it in the case p = 1.

Proposition 4.2. Let v be a Nikolskii-Wermer weight on $\mathbb{R}^{2d}$. $L^p_v(\mathbb{R}^{2d})$. is a Banach convolution algebra, i.e.,

$$\|F * G\|_{Lp,v} \quad < C_{p,v}\|F\|_{Lp,v}\|G\|_{Lp,v}$$

An application of Holder's inequality yields the assertion:

$$\begin{aligned}
\|(\tilde{F} * \tilde{G})v\|_{L^p} &= \|v(\tfrac{F}{v} * \tfrac{G}{v})\|_{L^p} \qquad (3.4)\\
&= \Big[\int_{\mathbb{R}^{2d}}\Big[\int_{\mathbb{R}^{2d}} \frac{|F(x,y)|}{v(x,y)}\frac{|G(x-z,y-t)|}{v(x-z,y-t)}dxdy\Big]^p v(z,t)^p dzdt\Big]^{1/p}\\
&= \Big[\int_{\mathbb{R}^{2d}}\Big[\int_{\mathbb{R}^{2d}} |F(x,y)||G(x-z,y-t)|\tfrac{v(z,t)}{v(x,y)v(x-z,y-t)}dxdy\Big]^p dzdt\Big]^{1/p}\\
&\le \Big[\int_{\mathbb{R}^{2d}}\Big[\Big(\int_{\mathbb{R}^{2d}}\Big(\frac{v(z,t)}{v(x,y)v(x-z,y-t)}\Big)^{p'}dxdy\Big)^{1/p'}\\
&\quad \Big(\int_{\mathbb{R}^{2d}}(|F(x,y)||G(x-z,y-t)|)^p dxdy\Big)^{1/p}\Big]^p dzdt\Big]^{1/p}\\
&\le C_{p,v}\|F\|_{L^p}\|G\|_{L^p} \le C_{p,v}\|\tilde{F}\|_{L^p_v}\|\tilde{G}\|_{L^p_v}.
\end{aligned}$$

Corollary 4.3. If s > d, then ($L^p{}_s$ ($\mathbb{R}^{2d}$),*) is a Banach convolution algebra.
Proof: A straightforward computation demonstrates that v_s is a Nikolskii-Wermer weight of L^p ($\mathbb{R}^{2d}$) for s > d.

Proposition 4.4.$H^p{}_s(\mathbb{R}^{2d})$is a Banach algebra with respect to pointwise multiplication for s>d.
Proof: Since the Bessel potential spaces $H^p{}_s(\mathbb{R}^{2d})$are weighted function spaces on theFourier transform side (Feichtinger, 1983) and (Tachizawa, 1994), then we have the result by taking a Fourier transform and applying the previous fact.

Corollary 4.5. The $Q^p{}_s(\mathbb{R}^{2d})$(Shubin class (Luef F. & Rahbani, 2011)), is a Banach algebra with respect to convolution and pointwise multiplication for s > d.

IV. CONCLUSION

In the current paper we introduce some important objects, convenience weight functions, for investigating one of the most important spaces in mathematics and specially in analysis, i.e., Banach spaces and show that through this useful tool we could have Banach algebras from some mentioned Banach spaces and employing new ways of algebraic structures. These algebraic spaces provide a width area of application in many branches of science and engineering such as signal processing, image processing with time frequency analysis, wavelet and frames theory, cryptography and etc.,.

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