Ilkogretim Online - Elementary Education Online, 2018; Vol 17 (Issue 2): pp. 1150-1154
http://ilkogretim-online.org

Non-Baire Proof Of Uniform Boundedness Theorem And Its Remarkable Applications

Abhijit Konch Assistant Professor Department of Mathematics Dhemaji College, Assam
abhijitkonch100@gmail.com

Abstract: In this paper our aim is to expound the importance of Non-Baire Proof of Uniform Boundedness theorem. We know that Baire's Category Theorem is directly applied to prove Uniform Boundedness Theorem. We present here, a proof of the Uniform Boundedness Theorem that does not require the Baire's Theorem in a similar fashion as proved by Alan D Sokal [1] but in a slightly different way.

Keywords: Uniform Boundedness, Open Mapping, Closed Graph.

1 Introduction:

The proving of three big theorems, known as the uniform boundedness theorem, the open mapping theorem, and the closed graph theorem, is the pinnacle of any first functional analysis course. All three rely on the completeness of some or all of the spaces involved, and their proofs are based on Baire's theorem (or the Baire category theorem), a topological conclusion. The open mapping theorem and the closed graph theorem are comparable in the sense that they may both be inferred from each other, available in most textbooks, one is proved first, starting with Baire's theorem, while the other is deduced afterwards. The uniform boundedness theorem is established on its own. We may also find publications in the mathematical literature that illustrate how to derive the uniform boundedness theorem from the closed graph theorem. With some limitations, the opposite is also true. These findings appear to be less well-known. The purpose of this note is to explore all of the possible equivalences in a clear and concise manner. In the case of Hilbert spaces, S. Kesavan [6] additionally proved that all three results are 'equivalent' to each other.

It is important to emphasize that all of these findings are distributed throughout the literature, and no claim to uniqueness in proof procedures is given. The proof of the uniform boundedness theorem from Baire's theorem is arguably the simplest of all of these proofs. However, multiple proofs of this result exist, in the sense that they do not utilize Baire's theorem, such as Hahn's using the "gliding hump" (also called "sliding hump") argument [5, Exercise 1.76]. In the context of Hilbert spaces, Halmos [3] also proves the uniform boundedness theorem without using Baire's theorem. We will show here a really simple demonstration of the uniform boundedness theorem that doesn't use Baire's theorem [due to Alan D. Sokal, 2011]. Here, we don't include direct proofs of the open mapping or closed graph theorems because they can be found in any functional analysis textbook.

Again it is important to note that the purpose of this article is not to minimize the significance of the Baire category theorem. Indeed, proofs of these statements using the Baire category theorem, which can be found in mainstream textbooks, are easier and more intuitive. The fact that they can be proved without using Baire's theorem, on the other hand, indicates that the completeness of the spaces involved is the foundation for these theorems.

2 Discussions and Result

We shall offer all the statement of the theorems for the sake of thoroughness of the exposition and to show the logical dependence of these results on each other. (For proof of the following theorem, see S. Kesavan [6])

Theorem 2.1: Each of the following statements implies the others.

(i) **The Closed Graph Theorem**: Let V and W be Banach spaces and let $T: V \to W$ be a linear map. If the graph of T is defined by

$$G(T) = \{(x, Tx): x \in V\} \subset V \times W$$

is closed in $V \times W$, then T is continuous.

(ii) **The Open Mapping Theorem**: Let V and W be Banach spaces and let $T: V \to W$ be a continuous linear map which is surjective. Then T is an open map, i.e. T maps open sets of V onto open sets of W.

(iii) **The Bounded Inverse Theorem**: Let V and W be Banach spaces and let $T: V \to W$ be a continuous linear bijection. Then T is an isomorphism, i.e. T^{-1} is also continuous.

(iv) **The Two Norms Theorem**: Let V be a vector space and let $\|.\|_1$ and $\|.\|_2$ be two norms on V. If V is a Banach space with respect to either norm and if there exists a constant $C > 0$ such that $\|x\|_1 \leq C\|x\|_2$ for every $x \in V$ then the two norms are equivalent.

(v) **Uniform Boundedness Theorem**: Let V be a Banach space and W be a normed linear space. Let $T_i: V \to W$ be a continuous linear map for each $i \in I$. If $\sup_{i\in I}\|T_i x\| < \infty)$ for each $x \in V$ then there exists a constant $C > 0$ such that $\|T_i\| \leq C$ for each $i \in I$.

The closed graph theorem was the beginning point in the article of S. Kesavan [6], and in order to finish the loop of the different implications, the argument that the uniform boundedness principle entails the closed graph theorem required the reflexivity of the target space. This consequence was demonstrated in the context of Hilbert spaces in the article referenced above. Ramaswamy and Ramasamy deal with the scenario where W is a reflexive Banach space.

Now we want to state the one of the pillars of Functional Analysis which is (UBT) and the proof given by the Alan D. Sokal [1].

To prove the Uniform Boundedness theorem, we need the following trivial result.

Lemma 2.2: Let T be a bounded linear operator from a normed linear space X to a normed linear space Y. Then for any $x \in X$ and $r > 0$, we have

$\sup_{x' \in B(x,r)} \|Tx'\| \geq \|T\| r$............(1)

Where $B(x,r) = \{x' \in X : \|x' - x\| < r\}$

Proof: For $\xi \in X$ we have

$\max\{\|T(x+\xi)\|, \|(x-\xi)\|\} \geq \frac{1}{2}[\|T(x+\xi)\| + \|T(x-\xi)\|] \geq \|T\xi\|$......(2)

where the $\geq$ uses the triangle inequality in the form $\|\alpha - \beta\| \leq \|\alpha\| + \|\beta\|$.

Now take the supremum over $\xi \in B(0,r)$.

Theorem 2.3 (Uniform Boundedness Theorem (Alan D Sokal)): Let $\mathcal{F}$ be a family of bounded linear operator form a Banach Space X to a normed linear space Y. If $\mathcal{F}$ is point wise bounded (i.e. $\sup_{T \in \mathcal{F}} \|Tx\| < \infty$ for all $x \in X$, then $\mathcal{F}$ is norm bounded. (i.e. $\sup_{T \in \mathcal{F}} \|T\| < \infty$)

Proof of theorem 2.3 : Suppose $\sup_{T \in \mathcal{F}} \|Tx\| = \infty$ and choose $(T_n)_{n=1}^{\infty}$ in $\mathcal{F}$ such that $\|T_n\| \geq 4^n$. Then set $x_0 = 0$ and for $n \geq 1$ use the lemma to choose inductively $x_n \in X$ such that $\|x_n - x_{n-1}\| \leq 3^{-n}$ and $\|T_n x_n\| \geq \frac{2}{3} 3^{-n} \|T_n\|$. The sequence (x_n) is a Cauchy, hence convergent to some $x \in X$; and it is easy to see that $\|x - x_n\| \leq \frac{1}{2} 3^{-n}$ and hence $\|T_n x\| \geq \frac{1}{6} 3^{-n} \|T_n\| \geq \frac{1}{6} (\frac{4}{3})^n \to \infty$.

The new proof of theorem 2.3 (UBT): Assume that $\sup_{T \in \mathcal{F}} \|Tx\| = \infty$ and choose $(T_n)_{n=1}^{\infty}$ in $\mathcal{F}$ such that $\|T_n\| \geq 6^n$. Then set $x_0 = 0$ and for $n \geq 1$ use the lemma 2.2 to choose inductively $x_n \in X$ such that $\|x_n - x_{n-1}\| \leq 5^{-n}$ and $\|T_n x_n\| \geq \frac{4}{5} 5^{-n} \|T_n\|$. The sequence (x_n) is a Cauchy, hence convergent to some $x \in X$;

Now if $m > n$ we have $\|x_n - x_m\| \le 5^{-(n+1)} + 5^{-(n+2)} + \cdots + 5^{-m}$

Keeping n fixed and letting $m \to \infty$ we deduce that

$$\|x_n - x\| \le \frac{5^{-n-1}}{1-\frac{1}{5}} = \frac{1}{4}5^{-n}$$

Then by the triangle inequality, we get

$\|T_n x\| = \|T_n x_n - T_n x_n + T_n x\| \ge \|T_n(x_n)\| - \|T_n(x - x_n)\| > \left(\frac{4}{5}5^{-n} - \frac{1}{4}5^{-n}\right)\|T_n\| = \frac{11}{20}5^{-n}\|T_n\| = \frac{11}{20}\left(\frac{6}{5}\right)^n \to \infty,$ a contradiction.

Apart from the Uniform Boundedness Theorem and other great theorems of Functional Analysis, we will look at two more conclusions included in most functional analysis textbooks that may be proved without utilising Baire's Theorem in this article. These are, without a doubt, consequence of UBT.

Theorem 2.4: Let $y = (\eta_i)$, $\eta_i \in C$ be such that $\sum \xi_i \eta_i$ converges for every $x = (\xi_i) \in c_0$ where $c_0 \subset l^\infty$ is a subspace of all complex sequences converging to zero. Then $\sum |\eta_i| < \infty$.

Theorem 2.5: Let X be a Banach space, Y a normed space and $T_n \in B(X, Y)$ such that $(T_n x)$ is Cauchy in Y for every $x \in X$. Then $(\|T_n\|)$ is bounded

To prove these results we have to use UBT and as proof of UBT is Baire-free so these results are also Baire-free.

Conclusion

As these results (involved in the above theorem 2.1) are interdependent and the Uniform Boundedness Theorem can be proved without appealing to Baire's Theorem, so these results are also Baire Free and we can use the Uniform Boundedness Theorem as a tool to prove these results.

References:

[1] A.D. Sokal, A really simple elementary proof of the uniform boundedness theorem, Amer. Math. Monthly, 118 no. 5, (2011) 450-452.

[2] E. Kreyszig, Introductory functional analysis with applications, John Wiley & Sons, New York, London, Sydney (1978)

[3] P.R. Halmos, A Hillbert Space Problem Book, Springer-Verlag, New York, 1974.

[4] R.E. Megginson: An Introduction to Banach Space Theory, Graduate Texts in Mathematics, vol 183. Springer, New York (1998)

[5] S. Kesavan, Functional analysis, TRIM, Hindustan Book Agency,52 (2009).

[6] S. Kesavan, A note on the grand theorems of functional analysis, Mathematics Newsletter, 27 no. 3, (2017) 188-191.

[7] W. Rudin, Real and Complex Analysis, 3rd edition, International Mc Graw-Hill, New York (1987).

[8] Walter Rudin, Functional Analysis, Second Edition, Tata McGraw Hill Edition.