Ilkogretim Online - Elementary Education Online, 2020; Vol 19 (Issue 4): pp. 7470-7477
http://ilkogretim-online.org
doi: 10.17051/ilkonline.2020.04.765149

The Study Of Inverse Laplace-Carson Integral Transform & Solution Of Linear Fractional Differential Equation

***Dr. Yaghvendra Kumawat , **Mohsin Turuk, ***Rashmi Modi**

Department of Mathematics, Vivekananda Global University, Jaipur-303012, Rajasthan.

Email. mohsin.turukengg@gmail.com , kumawat.yaghvendra@vgu.ac.in

ABSTRACT

The aim of present analysis is to obtain inverse Laplace-Carson integral transform formula with various condition and to obtain the solution of linear fractional differential equation. Illustrative example is given to demonstrate the validity, efficiency and applicability of the presented method. The solution obtained by the proposed method are in complete agreement with the solution available in the literature.

Keyword: - Laplace-Carson Integral, Fractional Differential Equation.

MSC 2010 Classification: 33E12, 34A08

1. INTRODUCTION

A scientific field known by the name of fractional calculus is considered to be as old as its counterpart called traditional or classical calculus. The former field is mostly about finding the derivatives and integrals of mathematical processes with arbitrary order whereas the traditional calculus does not enjoy such high degree of freedom. A tremendous growth has been observed in research work carried out in the field of fractional calculus. Large number of publications and new research journals based upon mathematical problems being studied in this field are the major interests of scholars [1–3]. Many such examples have been encountered in the past and in recent literature. For example, Abro in [4] has investigated the thermo-diffusion effects on unsteady-free convection flow, via fractional operator of Atangana-Baleanu for the governing mathematical equations in the presence of magnetic field. Various other mathematical models have been redefined in the framework of fractional calculus such as [5–12] and most of the references cited therein. Mathematicians, engineers and physicists cannot deny the fact that finding exact solutions for fractional order dynamical systems is practically impossible. To get an exact solution for the fractional order dynamical systems, integral transforms play an important role. The most popular and frequently used integral transform are Laplace transform followed by others including the

Fourier transform, Sumudu transform, Hankel transform, Mellin transform, Natural transform, Shehu transform, Elzaki transform, Aboodh transform, and the Mohand transform [13]. There is one more integral transform called the Laplace-Carson technique which, to the best of the authors' knowledge, has not been tested for getting the solutions of initial value problems defined by the Caputo operator. Thus, the present study is dedicated towards this goal. The theory of Laplace transform is referred to as operational calculus and now it becomes an essential part of the mathematical background which is required for engineers, mathematicians, physicists and other scientists. These methods provide an effective and easy means for the solution of many problems which are arising in the various fields of science and engineering. It is an important tool for solving ODEs and FDEs and has enjoyed much success in this realm. In the literature survey, there are various integral transforms which are extensively used in astronomy, physics as well as in engineering field. The integral transform method is also a useful method to solve the differential equations.

Laplace Transforms (Sneddon [14])**:**

Let F(z) be a function of z specified for z > 0. Then the Laplace transform of F(z), denoted by L{F(z)},

$$L[\,F(z)] = \int_{0}^{\infty} f(z).\, e^{-sz}.\, dz, \tag{1.1}$$

where we assume at present that the parameter s is real. Later it will be found useful to consider s complex. Laplace transform of F(z) is said to exist if the integral (i.e., eqn. no. 1.1) converges for some value of s; otherwise, it does not exist.

Mittag-Leffler function:

In the year 1903, the great Swedish mathematician Gosta Mittag-Leffler [15], introduced the function

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \tag{1.2}$$

where z is a complex variable and Γ(s) is a Gamma function, $\alpha \geq 0$. The Mittag- Leffler function naturally occurs as the solution of fractional order differential equations or fractional order integral equations. In 1905, Wiman [16], studied the generalization of $E_{\alpha}(z)$.

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}\ , (\alpha, \beta\ \in\ C\ ; Re(\alpha) > 0, Re(\beta) > 0). \tag{1.3}$$

which is known as Wiman's function or Generalized Mittag-Leffler function.

In order to perceive the main ambit of this research article, it is significant to know some basic information about non-integer order calculus and also some knowledge about the Laplace-Carson transform. In this regard, some relevant concepts are given below.

Definition 1 [17] The Laplace-Carson integral transform for a piecewise continuous function g(t) with exponential order P is defined over the set of functions:

$$A = \left\{g(t): \exists M\ , a_1 a_2 > 0, |g(t)| < M\, exp\left(\frac{|t|}{a_i}\right),\ if\ t \in\ (-1)^i\ \times [0, \infty]\right\}, \tag{1.4}$$

via the given integral

$$LC\ [g\ (t)] = G\ (p) = p \int_0^\infty exp\ (-pt)\ g\ (t)\ dt = p\ L\ \{g\ (t)\}, \tag{1.5}$$

where L shows the Laplace transform.

Inverse Laplace-Carson Transform

Now, we give the proof of Theorem 1-3 which are useful for finding the inverse Laplace-Carson transform

Theorem 1:

If $\mu, \sigma\ > 0, m\ \in R, |m| < p^{\mu}$, then we have the inverse Laplace − Carson transform formula

$$LC^{-1}\left[\frac{p^{\mu-\sigma+1}}{p^{\mu}+m}\right] = \left[t^{\sigma-1}\, E_{\mu,\sigma}\, (-mt^{\mu}\ )\right] \tag{1.6}$$

Proof: -

First, we take Laplace transform on the R.H.S of equation (1.6) and then applying equation (1.3), to get

$$LC\left[t^{\sigma-1}\, E_{\mu,\sigma}\, (-mt^{\mu}\ )\right] = p \int_o^\infty e^{-pt}\ t^{\sigma-1} \sum_{k=0}^{\infty} \frac{(-mt^{\mu})^k}{\Gamma(k\mu+\sigma)}\ dt$$

$$= p \sum_{k=0}^{\infty} \frac{(-m)^k}{\Gamma(k\mu+\sigma)} \int_o^\infty e^{-pt}\ t^{(\sigma+\ \mu k)-1}\ dt$$

$$= p \sum_{k=0}^{\infty} \frac{(-m)^k}{\Gamma(k\mu+\sigma)}\ \frac{\Gamma(k\mu+\sigma)}{p^{\sigma+\ \mu k}}$$

$$= \frac{1}{p^{\sigma-1}} \sum_{k=0}^{\infty} \left(\frac{-m}{p^{\mu}}\right)^{k}$$

$$= \frac{p^{\mu-\sigma+1}}{p^{\mu}+m}, \left|\frac{m}{p^{\mu}}\right| < 1$$

Then, the inverse Laplace-Carson transform is given by

$$LC^{-1}\left[\frac{p^{\mu-\sigma+1}}{p^{\mu}+m}\right] = \left[t^{\sigma-1} E_{\mu,\sigma}(-mt^{\mu})\right]$$

The proof is complete.

Theorem 2:

If $\mu \geq \sigma > 0, m \in R$ and $|m| < \frac{p^{\mu-\sigma}}{m}$ then the inverse Laplace-Carson transform formula

$$LC^{-1}\left[\frac{p^{1-\sigma(n+1)}}{(p^{\mu-\sigma}+m)^{n+1}}\right] = \left[t^{\mu(n+1)-1} \sum_{k=0}^{\infty} \frac{(-m)^{k}\binom{n+k}{k}}{\Gamma k(\mu-\sigma)+(n+1)\mu} t^{k(\mu-\sigma)}\right] \quad (1.7)$$

Proof: -

Similarly, to the proof of theorem 1, we take the Laplace transform of the right-hand side of equation (1.7) we get

$$LC\left[t^{\mu(n+1)-1} \sum_{k=0}^{\infty} \frac{(-m)^{k}\binom{n+k}{k}}{\Gamma k(\mu-\sigma)+(n+1)\mu} t^{k(\mu-\sigma)}\right]$$

$$= p\int_{o}^{\infty} e^{-pt}\ t^{\mu(n+1)-1} \sum_{k=0}^{\infty} \frac{(-m)^{k}\binom{n+k}{k}}{\Gamma k(\mu-\sigma)+(n+1)\mu} t^{k(\mu-\sigma)}\ dt$$

$$= p \sum_{k=0}^{\infty} \frac{(-m)^{k}\binom{n+k}{k}}{\Gamma k(\mu-\sigma)+(n+1)\mu} \int_{o}^{\infty} e^{-pt}\ t^{(\mu n+k\mu-k\sigma+\mu)-1}\ dt$$

$$= \frac{1}{p^{(n+1)\mu-1}} \sum_{k=0}^{\infty} \binom{n+k}{k} \left(\frac{-m}{p^{\mu-\sigma}}\right)^{k}$$

$$= \frac{p^{1-\sigma(n+1)}}{(p^{\mu-\sigma}+m)^{n+1}}$$

Then the inverse Laplace-Carson transform is given by

$$LC^{-1}\left[\frac{p^{1-\sigma(n+1)}}{(p^{\mu-\sigma}+m)^{n+1}}\right] = \left[t^{\mu(n+1)-1} \sum_{k=0}^{\infty} \frac{(-m)^k \binom{n+k}{k}}{\Gamma k(\mu-\sigma)+(n+1)\mu} t^{k(\mu-\sigma)}\right]$$

The proof is complete.

Theorem 3:

If $\mu \geq \sigma, \mu > \rho, m \in R, |m| < p^{\mu-\sigma}$ and $\left|\frac{lp^{-\sigma}}{p^{\mu-\sigma}+m}\right| < 1$ then we have inverse Laplace-Carson transform formula

$$LC^{-1}\left[\frac{p^{\rho-\sigma+1}}{p^{\mu-\sigma}+m+lp^{-\sigma}}\right] = \left[t^{\mu-\rho-1} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-l)^n(-m)^k \binom{n+k}{k}}{\Gamma(k(\mu-\sigma)+(n+1)\mu-\rho)} t^{k(\mu-\sigma)+n\mu}\right] \quad (1.9)$$

Proof: -

We take the Laplace-Carson transform on the right-hand side of equation (1.9), by using the definition of gamma function and using series expansion, we get

$$LC\left[t^{\mu-\rho-1} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-l)^n(-m)^k \binom{n+k}{k}}{\Gamma(k(\mu-\sigma)+(n+1)\mu-\rho)} t^{k(\mu-\sigma)+n\mu}\right]$$

$$= p\int_{o}^{\infty} e^{-pt}\ t^{\mu-\rho-1} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-l)^n(-m)^k \binom{n+k}{k}}{\Gamma(k(\mu-\sigma)+(n+1)\mu-\rho)} t^{k(\mu-\sigma)+n\mu}\ dt$$

$$= p \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-l)^n(-m)^k \binom{n+k}{k}}{\Gamma(k(\mu-\sigma)+(n+1)\mu-\rho)} \int_{o}^{\infty} e^{-pt}\ t^{(k\mu-k\sigma+n\mu+\mu-\rho)-1}\ dt$$

$$= \sum_{n=0}^{\infty} \frac{(-l)^n}{p^{(1+n)\mu-\rho-1}} \frac{1}{\left(1+\frac{m}{p^{\mu-\sigma}}\right)^{n+1}}$$

$$= \frac{p^{(\rho-\sigma+1)}}{(p^{\mu-\sigma}+m)} \sum_{n=0}^{\infty} \left[\frac{-lp^{-\sigma}}{(p^{\mu-\sigma}+m)}\right]^n$$

$$= \frac{p^{(\rho-\sigma+1)}}{(p^{\mu-\sigma}+m+lp^{-\sigma})}$$

Then, the inverse Laplace-Carson transform is given by

$$LC^{-1}\left[\frac{p^{\rho-\sigma+1}}{p^{\mu-\sigma}+m+lp^{-\sigma}}\right] = \left[t^{\mu-\rho-1} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-l)^n(-m)^k \binom{n+k}{k}}{\Gamma(k(\mu-\sigma)+(n+1)\mu-\rho)} t^{k(\mu-\sigma)+n\mu}\right]$$

The proof is complete.

Illustrative example:

In this section, we shall illustrate the applicability of the inverse fractional Laplace-Carson transform method to some of the linear differential equations.

Example

Consider the following linear fractional initial value problem (Odibat et al.2008)),

$^{C}D^{\mu}\, g(t) = g(t) + 1,\ 0 < \mu \leq 1$ (2.0)

subject to initial condition g (0) = 0, has the solution

$$t^{\mu+1} E_{\mu,\mu+1}(t^{\mu})$$

Applying the Laplace-Carson transform to both sides of the equations (2.0), and using example 1 from P. Kumar & S. Qureshi [18][page no.62, eqn. 15], we get

$LC\left[\,^{C}D^{\alpha}\, g(t)\right] = LC\,[g(t)] + LC\,[1]$

$$p^{\mu} G(p) = G(p) + 1$$

$$(p^{\mu} - 1)\ G\ (p) = 1$$

$$G\ (p) = \frac{1}{p^{\mu} - 1} \tag{2.1}$$

Now, by using the result of theorem 1 and taking the inverse Laplace-Carson on both side of equation (2.1), and by putting $\sigma = \mu + 1$ & $m = -1$ in theorem 1, we get

$$LC^{-1}\left[\frac{p^{\mu-(\mu+1)+1}}{p^{\mu}-1}\right] = t^{\mu+1-1}E_{\mu,\mu+1}(t^{\mu})$$

Therefore, the exact solution of the problem can be obtained as

$$g(t) = t^{\mu}E_{\mu,\mu+1}(t^{\mu})$$

Conclusion:

In this theorem, a new method called the inverse fractional Laplace-Carson transform method have been successfully applied to linear fractional differential equation. We proved three theorems related to this method. The resolution of some example shows that the inverse Laplace-Carson transform method is more powerful and efficient for finding exact solutions of linear fractional differential equations.

References:

1. Sekerci, Y., & Ozarslan, R. (2020). Oxygen-plankton model under the effect of global warming with nonsingular fractional order. Chaos, Solitons and Fractals, 132, 109532.
2. Sekerci, Y., & Ozarslan, R. (2020). Dynamic analysis of time fractional order oxygen in a plankton system. The European Physical Journal Plus, 135(1), 88.
3. Sekerci, Y., & Ozarslan, R. (2019). Mathematical modelling of plankton–oxygen dynamics in view of non–singular time fractional derivatives. Physica A: Statistical Mechanics and its Applications, 12391242.
4. Abro, K.A. (2020). A fractional and analytic investigation of thermo-diffusion process on free convection flow: an application to surface modification technology. The European Physical Journal Plus, 135(1), 31.
5. Atangana, A., & Qureshi, S. (2019). Modeling attractors of chaotic dynamical systems with fractal-fractional operators. Chaos, Solitons & Fractals, 123, 320-337.
6. Ghanbari, B., Yusuf, A., & Baleanu, D. (2019). The new exact solitary wave solutions and stability analysis for the (2+ 1) (2 + 1)-dimensional Zakharov-Kuznetsov equation. Advances in Difference Equations, 2019(1), 1-15.
7. Arqub, O.A., & Al-Smadi, M. (2020). An adaptive numerical approach for the solutions of fractional advection–diffusion and dispersion equations in singular case under

Riesz's derivative operator. Physica A: Statistical Mechanics and its Applications, 540, 123257

8. Qureshi, S., & Aziz, S. (2019). Fractional modeling for a chemical kinetic reaction in a batch reactor via nonlocal operator with power law kernel. Physica A: Statistical Mechanics and its Applications, 123494.
9. Abu Arqub, O. (2019). Application of residual power series method for the solution of time- -fractional Schrodinger equations in one-dimensional space. ¨ Fundamental Informaticae, 166(2), 87-110.
10. Qureshi, S., & Memon, Z.-U.-N. (2020). Monotonically decreasing behavior of measles epidemic well captured by Atangana–Baleanu–Caputo fractional operator under real measles data of Pakistan. Chaos, Solitons & Fractals, 131, 109478.
11. Atangana, A., & Khan, M.A. (2019). Validity of fractal derivative to capturing chaotic attractors. Chaos, Solitons & Fractals, 126, 50-59.
12. Owolabi, K.M., & Atangana, A. (2019). Mathematical analysis and computational experiments for an epidemic system with nonlocal and nonsingular derivative. Chaos, Solitons & Fractals, 126, 41-49.
13. Qureshi, S., & Kumar, P. (2019). Using Shehu integral transform to solve fractional order Caputo type initial value problems. Journal of Applied Mathematics and Computational Mechanics, 18(2), 75-83.
14. L. N. Sneddon, The Use of Integral Transforms, Tata McGraw-Hill, New Delhi,1979.
15. G. M. Mittag-Leffler, sur la nouvelle function $E_\alpha(z)$, C. R. acad. Sci. Paris 137 (1903) 554-558.
16. A. Wiman, Uber den fundamental Satz in der Theorie der Funktionen $E_\alpha(x)$, Acta Math. 29 (1905) 191-201.
17. Maitama, S., & Zhao, W. (2019). New integral transform: Shehu transform a generalization of Sumudu and Laplace transform for solving differential equations. International Journal of Analysis and Applications, 17(2), 167-190.
18. P. Kumar & S. Qureshi, Laplace-Carson integral transform for exact solutions of non-integer order initial value problems with Caputo operator. Journal of applied mathematics and computational mechanics 2020,19(1),57-66.