Ilkogretim Online - Elementary Education Online, 2020; 19 (3): pp. 1321-1336
http://ilkogretim-online.org.tr
doi:10.17051/ilkonline.2020.729658

The role of concept images in solving geometric word problems[1]

Elif Karatağ Şahin, *Ministry of Education,* Turkey, *karatagelif@gmail.com* ORCID: 0000-0002-3150-5218
Hilal Gülkılık, *Gazi University,* Turkey, *ghilal@gazi.edu.tr* ORCID: 0000-0002-2664-3288
Hasan Hüseyin Uğurlu, *Gazi University,* Turkey, *hugurlu@gazi.edu.tr* ORCID: 0000-0002-9900-6634

Abstract. The purpose of this study was to investigate how eighth-grade students were affected by their concept images of mathematical concepts in solving geometric word problems. Five eighth-grade students from a public middle school participated in the study in the second semester of the 2016-2017 academic year. The data were obtained through semi-structured preliminary interviews to identify the students' concept images of basic geometry concepts and task-based interviews in which thirteen word problems involving these concepts were presented. The data were analyzed using descriptive analysis techniques. The findings were classified within three themes as the role of concept images in understanding, modeling, and mathematical analysis stages of word problem solving. Concept images affected the students in constructing a situation model for the problem, transforming the situation model into a mathematical model, and performing mathematical operations to obtain a correct solution.

Keywords: Problem solving, geometric word problems, concept image, concept definition

Received: 28.04.2019	Accepted: 21.11.2019	Published: 15.06.2020

INTRODUCTION

For mathematics teachers and researchers, and in general for most people, doing mathematics means solving problems (Krulik & Rudnick, 1988). Problem solving is one of the most difficult skills that mathematics education aims to develop (Stacey, 2005) and it has an important role in modeling "the act of doing mathematics in the real world" (Van De Walle, Karp, & Bay-Williams, 2010, p. 13). In fact, one of the main reasons for teaching mathematics to students in school is to provide them with the competencies they need to use mathematics in real-life situations (Verschaffel, De Corte, & Lasure, 1994). Although school mathematics provides students with useful tools to find solutions to problems in real life, they have difficulty selecting and using such mathematical tools appropriately (Muller & Burkhardt, 2007). One of the tools that can be used in mathematics education is word problems. Word problems allow students to have a chance to practice for real-life problem situations, motivate students to understand mathematical concepts, and contribute to their creative and critical thinking skills (Chapman, 2006).

Verschaffel, Greer, and De Corte (2000) stated that "word problems can be defined as verbal descriptions of problem situations wherein one or more questions are raised the answer to which can be obtained by the application of mathematical operations to numerical data available in the problem statement" (p. ix). They point out that word problems to be used in mathematics teaching should include realistic and imaginable situations. Researchers (e.g., Galbraith & Stillman, 2001; Reed, 1999) categorize word problems in different ways. In this study, the focus is on geometric word problems involving real-life situations that students are likely to encounter in the future.

Word problems may be conceived of "as a specific—some might say: over-simplified—type of mathematical modeling problems" (Verschaffel, Schukajlow, Star, & Van Dooren, 2020, p.2). In this vein, the characterization of the process of solving mathematical modeling problems can also be used to describe the process of solving word problems (Verschaffel, Greer, & De Corte, 2002, see Figure 1). Kertil, Çetinkaya, Erbaş, and Çakıroğlu (2016) support this idea in their analysis of different approaches to mathematical modeling in the mathematics education literature.

[1]This article was produced from the first author's master's thesis completed under the supervision of the third author. The second author made a substantial contribution to the article. An earlier version of this article was presented at the International Learning, Teaching and Education Research Congress in Turkey, 2018.

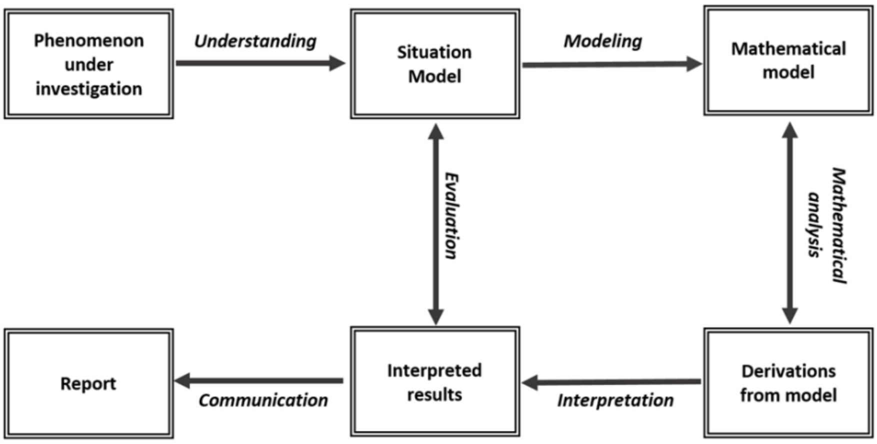

FIGURE 1. *The process of modeling (Verschaffel et al., 2002, p. 258)*

According to Verschaffel et al. (2002), the first competency expected from students in solving word problems is that they can understand the problem situation by its reality. Students construct an appropriate situation model reflecting an understanding of the components of the problem situation and generate this situation model by using external representations such as pictures, drawings, and verbal expressions. Students transfer the structures in the situation model to mathematical language and explain the model with mathematical tools in the stage of mathematical modeling. In the stage of mathematical analysis, students derive some results by performing computational work with the components included in their mathematical models. They then interpret the results and evaluate whether the interpreted results are correct and reasonable. In the last stage, students communicate their solution.

Researchers (e.g., Kilpatrick, Swafford, & Findell, 2001; Mayer & Wittrock, 2006; Schoenfeld, 1985) state that students' knowledge of mathematical concepts will affect their problem-solving proficiency. Strong conceptual knowledge helps a student not only select the appropriate operation in problem solving but also check whether the result of the operation is reasonable (Hiebert & Lefevre, 1986). In addition to procedural knowledge, heuristic methods, metacognition, and positive task-related affects, students need well-structured conceptual knowledge to solve word problems (Arslan & Altun, 2007; De Corte, Greer, & Verschaffel, 1996; Schoenfeld, 1992). Concept image and concept definition are among the prominent theoretical frameworks in the mathematics education literature that focus on students' knowledge of mathematical concepts (Bingölbali & Monaghan, 2008).

Concept Image and Concept Definition

A concept image is "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152). In a concept image, in addition to the mental pictures related to the concept, there may be some true or false features attributed to the concept (Vinner, 1983). For example, a student may have a perception that the result of multiplication is always greater than the multipliers. Such a perception shows that the student overgeneralizes the rule of multiplication that is valid only in the case of positive numbers and constructs incorrect knowledge in his or her image of the concept of multiplication (Graeber & Campbell, 1993). Since some incorrect ideas in concept images may be shaped as a result of overgeneralization or overspecialization, students' images of a mathematical concept may also contain misconceptions (Bingölbali, 2016; Viholainen, 2008).

Depending on the nature of a problem situation that a student deals with, only some parts of the concept image may become activated in the student's mind (Vinner, 1983). Tall and Vinner (1981) explain the relevant part of a concept image that becomes activated in a certain period of time with the term evoked concept image. For example, when a student is asked a question about quadrilaterals and if only convex quadrilaterals come to his or her mind even though the question requires considering concave quadrilaterals, the student's evoked image may lead to difficulty in addressing the question (Bingölbali, 2016). Tall and Vinner (1981) state that contradictory parts of a concept image may be evoked in students' minds in different contexts. According to them, if these conflicting parts of the image come to a student's mind simultaneously while he or she is trying to solve a problem, it may cause confusion or cognitive conflict.

On the other hand, Tall and Vinner (1981) describe the concept definition as a "form of words used to specify that concept" and emphasize that "a personal concept definition can differ from a formal concept definition" (p. 152). Since a concept image includes various elements, the term coherence of a concept image can be used to refer to the internal organization of these elements (Viholainen, 2008). In particular, Yanik (2014) determines the level of the coherence of students' concept images "by the consistency between their concept images and their concept definitions" (p. 46). Although formal definitions of mathematical concepts are given to students in teaching settings, it is observed that students tend to use concept images more when solving problems (Vinner, 1983). However, concept images may not always correspond to the formal definition of a concept or may be completely incorrect (Tall & Vinner, 1981). Vinner and Dreyfus (1989) argue that there is often a conflict between students' concept images and the definition of the concept presented by the teacher, and in such moments, student behaviors differ from what the teacher expects. Similarly, Rösken and Rolka (2007) reveal that in the case of a discrepancy between a concept definition and a concept image, the concept image will prevail, and this situation will affect students' problem solving. Therefore, identifying how students are affected by concept images while solving geometric word problems that involve many cognitive activities may allow us to evaluate their experiences in such situations in detail.

Students' Experience with Word Problems

A growing body of literature has examined students' problem-solving performances for word problems. In particular, researchers examined students' ability to use real-life knowledge when solving word problems. Greer (1993), for example, analyzed how students aged 13-14 solved standard word problems that could be solved by means of the most obvious arithmetic operations with the numbers in the problem and problematic word problems that could be solved by taking into account real-world knowledge. He found that the performance of students for standard word problems was high, while their performance for the latter problems was low. In a parallel study conducted with 75 students, Verschaffel et al. (1994) found that many of the students' answers to problematic word problems were far from reality. Similarly, Çelik and Güler (2013) found that sixth-grade students tried to solve real-life problems just like routine problems[2], tended to use all the numbers given in the problem, and chose incorrect operations. Bayazit (2013), who examined the approaches, strategies, and models utilized by seventh- and eighth-grade students in solving real-world problems, revealed that a great majority of the students applied their procedural knowledge without thinking about the problem situations and produced unrealistic responses. Although some students constructed models of the problem situations, they used these models as a part of routine procedures instead of a conceptual tool (Bayazit, 2013). Students ignore real-life situations and they perform mathematical operations with only given numbers, without considering the meaning of solutions they reach in real life (Verschaffel, Greer, Van Dooren, & Mukhopadhyay, 2009).

The reason why students cannot solve these problems in the way of producing realistic answers has been the subject of many further studies. The necessity for students to transfer knowledge from mathematics to daily life and the need to create original models and to make logical inferences for the solution are among the factors affecting the process (Chacko, 2004). Karataş and Güven (2010), who sought to examine the abilities of ninth- and eleventh-grade students to solve realistic word problems and to make comparisons between these two classes to

[2]Suydam (1987) indicates that routine problems can be solved by the application of a standard method in contrast to non-routine problems that require more cognitive work than applying a known procedure (as cited in Asmana & Markovits, 2009).

determine the development of students in solving realistic word problems, emphasized the importance of a similar point. The results of their research showed that students' ability to construct a mathematical model representing a real-life situation is a determining factor in their proficiency.

In his research with 161 fifth-grade students, Palm (2008) investigated how the students used their real-life knowledge in solving problems and determined how they transferred their solutions to real life. The students were asked the same problem situations in two different ways, with and without more authentic variants. The authenticity of the problem was described as the concordance between the problem "that includes a description of a 'real' out-of-school situation" and "the actual real life situation" (p. 39). For example, the problem "360 students will take a school trip by bus. Since there will only be 48 students on each bus, how many buses are needed?" was a less authentic problem (p. 43). The more authentic version of the same problem was:

> All students in our school will go on a trip on May 15. You are in charge of the organization and have decided to take the students in our school by bus. You have realized there are 360 students in our school. You have visited a company to rent a bus, and the company officials have stated that the maximum number of students on a bus should be 48. How many buses are required to go on the trip? (Palm, 2008, p. 43)

Palm (2008) indicated that task authenticity had an impact on students' use of real-world knowledge in their solutions to the problems. Hoogland, Koning, Bakker, Pepin, and Gravemeijer (2018), who used a different method to reduce the difficulties of students in solving word problems, gave the problems to the students with depictive representations of the problem situations. It was observed that the students scored significantly better on image-rich problems, but with a very small effect size. Gkoris, Depaepe, and Verschaffel (2013) examined the issue from a different perspective and performed a comparative analysis of word problems in the mathematics textbooks of fifth-graders in Greece. The results revealed that the word problems in the old and new fifth-grade mathematics textbooks could be solved by converting the data directly to mathematical operations without the need for realistic thinking.

Although extensive research has been carried out on word problems, there has been little analysis focusing on geometric word problems (Ulusoy & Argun, 2019). Students need to understand and use different representations of geometric concepts to solve geometric word problems efficiently (Mesquita, 1998). Additionally, most geometric word problems involve the characteristics of arithmetic word problems (Wong, Hsu, Wu, Lee, & Hsu, 2007). Despite this importance, to the best of our knowledge no research has examined the traces of concept images in the solution of geometric word problems. A student who tries to solve a geometric word problem should be able to use concept images of mathematical concepts effectively while constructing a situation model appropriate to the problem, transferring this model to a mathematical model, and generating a solution with the help of the mathematical model. Because students use concept images rather than concept definitions while doing mathematics (Vinner, 1983), a detailed examination of how concept images affect students during geometric word problem solving is important for understanding the experience of students in solving these problems. In Turkey, eighth-grade students will take the High School Entrance Exam, which includes geometry word problems. The results of the current research may help us understand the conceptual foundations of the difficulties experienced by eighth-grade students in solving geometric word problems. The research problem was specified as follows: How do eighth-grade students' concept images of mathematical concepts affect the students in solving geometric word problems?

METHODS

The research was designed as a case study. A case study is "an empirical inquiry that investigates a contemporary phenomenon (the 'case') in depth and within its real-world context, especially when the boundaries between the phenomenon and context may not be clearly evident" (Yin, 2014, p. 16). Researchers should design case studies if they are "interested in insight, discovery and interpretation rather than hypothesis testing" (Merriam, 2009, p. 42). Therefore, it was thought that the case study method would foster a detailed and in-depth analysis of how students' concept images affect the process of solving geometric word problems.

Participants

The study was conducted with five eighth-grade students enrolled in a public middle school. Throughout the study, pseudonyms are used instead of students' real names. Since the first author (the researcher henceforward) was also the mathematics teacher of the participants, she knew the participants closely through both in-class and out-of-class observations. In the study, the purposeful sampling method (Patton, 2002) was used in identifying the participants, each of whom was considered as a case. One of the participants, Rukiye, was a student with high academic achievement in mathematics. She actively participated in class activities, had an advanced sense of self-confidence, and could express herself easily. Another participant, Nuh Can, was a student with average academic achievement in mathematics. Although he showed active participation in the class, it was observed that he was not able to develop a permanent understanding of what he had learned in mathematics lessons. Servet, who remained quiet both in the class and among his friends, was a student who was not able to achieve the expected success in the mathematics class. Seynur, on the other hand, was a student who showed below-average mathematics performance in class and exhibited characteristics such as lack of attention, inability to perform appropriate logical reasoning, and low self-confidence. Hanife, the last participant, was a student who participated in class activities only with the urging of her teachers and had average academic achievement in mathematics.

Data Sources and Procedure

The word problems that were presented to the participants in the study included several basic geometric concepts. First, a review and discussion of the middle school mathematics curriculum and relevant literature were analyzed to identify these geometric concepts. The concepts of similarity, slope, area, surface area, angle, translation, rotation, and reflection were selected. Semi-structured interviews were then conducted with the participants. In the interviews, an interview form consisting of 29 open-ended questions to identify the students' concept images of these concepts was used. Following the interviews, researchers posed 13 geometric word problems based on real-life situations generated on the basis of these concepts.

Two geometry word problems were generated based on each geometric concept of similarity, slope, angle, and area, while this number was four with the concept of surface area. Only one problem was generated based on concepts of translation, rotation, and reflection. Solving these problems required much greater cognitive effort compared to applying one or more mathematical operations to the numbers in the problem text. Taking Palm's (2008) findings into consideration, researchers increased the authenticity of the problems. Students had to imagine the context in which the problem was designed to create an appropriate situation model for the problem (e.g., a number of square pyramid-shaped camping tents or two castle windows with different heights from the ground and in the same direction). They would then have to identify the necessary mathematical concepts and operations to interpret the situation model mathematically (e.g., using the concept of surface area to determine the amount of fabric required for the camping tents or establishing a relationship between the positions of windows and the concept of similarity).

Necessary corrections and changes were made according to field experts' opinions (two mathematics education professors) in generating geometric word problems and preparing interview protocols. In this context, to ensure intelligibility, visual elements were added to some of the problems, or the sentences in the problem statements were written more clearly. The 13 geometric word problems were presented to 11 eighth-grade students enrolled in a different school in the pilot study. The parts in the interview form that the students had difficulty in understanding were identified, and the situations that caused misunderstandings were changed.

Following the pilot study, one-to-one task-based interviews (Goldin, 2000) including the geometric word problems were conducted with the participants. Four interviews were conducted with each student once or twice a week. Since the researcher was also the teacher of the participants, the interviews were conducted with the students outside of school hours and in informal environments. The participants were reminded that the interviews would never affect their academic assessments so that they could answer the interview questions as comfortable as possible. All the drawings and expressions of the students were recorded with a video camera. Each interview lasted 50-60 min.

Data Analysis

In the first stage of the data analysis, the students' solutions of each geometry word problem were coded by using descriptive analysis based on the stages described by Verschaffel et al. (2002). Since the purpose of the research was to determine how students' concept images affected them in generating solutions for geometric word problems, data analysis was limited to the first stages of understanding, modeling, and mathematical analysis. Following the coding, the categories were formed, and themes were identified by revealing the relationships between the categories. In order to ensure the reliability of the coding, a different mathematics education researcher checked 10% of the codes. According to her feedback, the names of some codes were changed, or new codes were included to better emphasize the characteristics of the codes. In the second stage, traces of students' concept images were sought within the themes, and codes were formed about how the concept images affected this process. It was acknowledged that there might be many other factors that would affect the process of solving geometric word problems. Therefore, while creating the categories, the focus was on the sections in which the effects of concept images come to the fore during the solution process.

Several techniques were used to ensure the trustworthiness of the study. Expert opinions were consulted at each stage of the study and a pilot study was conducted. Observations and written documents belonging to students were used to triangulate the data collected in the interviews.

RESULTS

Findings revealing the traces of concept images in the solution of geometric word problems were categorized within three main themes. In this vein, the role of concept images in the stages of i) understanding the problem, ii) mathematical modeling of the problem situation, and iii) mathematical analysis of the model will be presented, respectively.

Concept Images in Understanding the Problem

Understanding the problem is critical in terms of being the first stage when a student perceives the *problem*. In this stage, students should be able to determine what information is given in the problem and what they are being asked to find or do (Polya, 1957). The findings showed that concept images were critical in facilitating the understanding of the problem. For example, in the first step of the problem called "We Are Going to Camp," students were asked to determine the amount of fabric required to design a square pyramid-shaped tent in a specific size. It was observed that the students who tried to understand the problem focused primarily on creating a proper visual representation of the square pyramid-shaped camp tent. From the interview with Servet, the following dialogue, which occurred after he read the problem, can be given as an example of the difficulties experienced by students whose evoked concept images were inappropriate for the problem situation:

Researcher:	The scouts going to have these tents built. How much fabric do they need to get all the tents built? They want you to help them with that. How can we find it? What do you think?
Servet:	I will draw a square pyramid first [He draw a square prism; see Figure 2].
Researcher:	Will our square pyramid-shaped tents be like this, Servet?
Servet:	Yes.

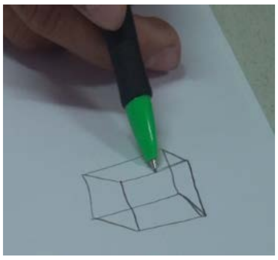

FIGURE 1. *Servet's drawing of a pyramid*

Since Servet did not take any action after reading the problem, the researcher explained the problem situations in her own words. Servet drew a prism instead of a pyramid, which shows that his concept image of pyramid coincides with the concept of prism. Since concept images are nonverbal and implicit structures (Vinner, 1983), the researcher asked different questions about these two geometric concepts, and the dialogue continued as follows:

Researcher:	Ok, can you draw me a pyramid? For example, how do you draw a square pyramid?
Servet:	[Keeps silent for a few seconds and draws the small prism in Figure 3 without responding to the question]
Researcher:	Now, you've drawn a square pyramid on one side, a square prism on the other. What's the difference between them?
Servet:	[Keeps silent]
Researcher:	One of them is a bit bigger than the other. Is there any difference between them except for their sizes?
Servet:	No, there is not.

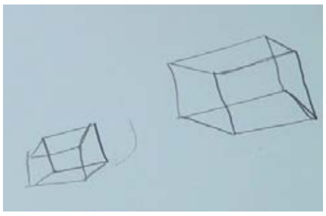

FIGURE 2. *Servet's comparison of a square pyramid and square prism*

When Servet first read the problem, he probably could not imagine a square pyramid-shaped camp tent. When the interviewer asked him to draw a square pyramid separately from the context of the problem, he drew a square prism again. Servet had problems in distinguishing the concepts of pyramid and prism while he was trying to understand the problem. However, like the other students, he was able to draw pyramids by specifying the vertex points while defining the concept of pyramid during the preliminary interviews. The inconsistency between Servet's concept definition and concept image seemed to make it difficult to create a situation model for the problem. Later in the interview, Servet was able to draw the square pyramid when the researcher stated that different geometric objects should look different and helped him to draw an appropriate representation of the concept. Other students, except Rukiye and Hanife, also drew prisms instead of pyramids while solving the problem. These students, whose evoked concept images of square pyramid were inappropriate, had more difficulty in constructing the situation model of the problem compared to the two students who had appropriate concept images of the concept.

A similar finding was revealed in the geometric word problem called "School Trip." The problem was presented through a story in which the students were asked to help stonemasons who were restoring the Rumelian Castle with the construction of windows. A photo of the castle and the image of a ladder were provided. The students had to work on windows with different heights from the ground and in the same direction. When we say that the windows will be in the same direction, we mean that the corresponding sides of the windows will be in the same direction. Students had problems in placing the windows, especially because of their concept image of direction, and they were able to create the situation models as a result of interventions by the researcher. Nuh Can's statements for this problem are presented below:

Researcher: What is the stonemason doing in this window?
Nuh Can: He puts a 3-meter-long ladder at the castle's window.
Researcher: So, how does he put the ladder at the window?
Nuh Can: [Draws the left image in Figure 4 without responding to the question; he writes "2m" to show that the window was 2 meters above the ground]
Researcher: Let's assume there is another window, and these two windows are in the same direction.
Nuh Can: Yes, in the same direction [Draws the right image in Figure 4; although he draws a window to be at the same height as the height of the first window, he writes "6 m" on it].

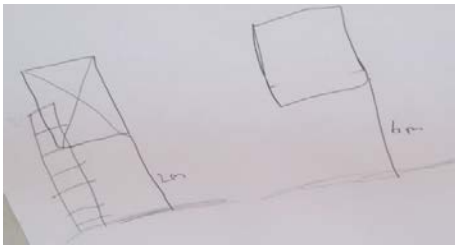

FIGURE 3. *Nuh Can's drawing of windows with different heights but in the same direction*

When Nuh Can was reminded that he had to draw windows with different heights, he said "Yes, but it says they are in the same direction." His evoked concept image of the concept of direction seemed to involve the idea that objects will only be in the same direction when placed horizontally next to each other. Similar to Servet, Nuh Can could draw the windows in the vertical direction when the researcher used interventions including several questions about objects in same directions. However, these interventions prolonged the process of creating an appropriate situation model for the problem and influenced the solution.

On the other hand, concept images that were mathematically appropriate accelerated students' understanding of geometric word problems significantly. For example, Rukiye, who had appropriate concept images of geometric transformations, could fulfill the instructions quickly and accurately in the "Tile Master" problem. In the problem, the students were asked to help a craftsman who wanted to decorate a building with tiles in the placement of the tiles. One of the tasks of the craftsman was to place the tiles at regular intervals by sliding them. Rukiye restated the problem in her own words as follows and created an appropriate situation model:

Rukiye: It says there is going to be a tile decoration, and there are three ideas. Mr. Adem wants me to help him decide which of these three ideas he should choose.
Researcher: What was the first idea? How will you place the tiles?
Rukiye: He will place the tiles at regular intervals. He will slide the given tile 5 cm. Let me draw the tiles first [She translated all corners of the first tile by five centimeters while drawing the second tile; see Figure 5].
Researcher: Do you consider all the corners of the tile?
Rukiye: Yes, we should look at all the corners of the shapes when translating or reflecting.

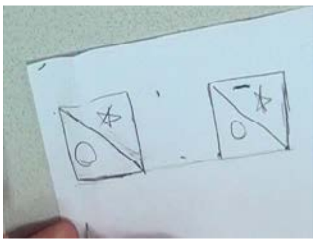

FIGURE 4. *Rukiye's drawings in translating a tile*

While translating the tile, it was observed that Rukiye paid attention to the fact that each corner point is affected by the translation. Rukiye has a concept image of translation appropriate for the middle school level, although she understands translation as a motion. This can be understood from the following dialogue during the preliminary interviews about the concept:

Researcher:	Can you give me an example of translation?
Rukiye:	Let's do this on the coordinate plane. For example, let's assume there is a square here. [Draws a square with one of the corner points (3,6)]. Let its coordinate be (3,6). Let's translate it 3 units left and 2 units down. Since it is 3 units to the left, we will apply it on the x-axis and do subtraction. When it comes right, we will do addition. When I subtract 3 on the x-axis, the new abscissa will be 0. The new ordinate will be 4.

The above statements, which give clues about the details of Rukiye's image of the concept of translation, confirm that the concept image helped her in the stage of constructing a situation model for understanding the problem. In the next section, the findings about students' mathematical models, which were based on their situation models, will be presented.

Concept Images in Mathematical Modeling

Following the creation of a situation model for geometric word problems, students begin to interpret these models mathematically and proceed to the next stage of the solution (Berry & Houston, 1995). They try to identify the mathematical objects that will best reveal the relationship between variables in the situation model (Abrams, 2001). The findings showed that the students' concept images guided the process of identifying the mathematical concepts they needed to solve the problems. For example, in the "Art Class" problem, students were asked to design a cardboard lampshade in accordance with the given measurements. Immediately next to the problem text, a picture of a lampshade was provided (see Figure 6). With the help of the questions that she asked when Nuh Can first read the problem, the researcher found that Nuh Can knew what a lampshade was and how to use a lampshade in daily life. However, Nuh Can's following statements were an example of how incorrect concept images of geometric objects affect mathematical modeling:

Researcher:	What shape does the lampshade look like?
Nuh Can:	It looks like a trapezoid [He draws a trapezoid but adds circles on the top and bottom of it; see Figure 6].
Researcher:	Are there circles on the top and bottom of a trapezoid?
Nuh Can:	No, it might look like a cylinder.
Researcher:	Can you show me a cylinder?
Nuh Can:	[Draws a cylinder next to the trapezoid without responding to the question; see Figure 6]
Researcher:	Do the lampshade and the cylinder look like each other?
Nuh Can:	No.

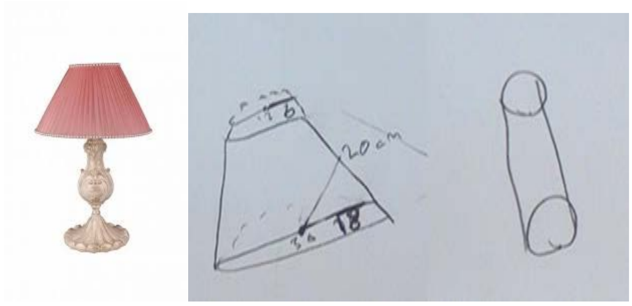

FIGURE 5. *Picture of lampshade given in the problem text and Nuh Can's drawings for the lampshade*

Later in the interview, he continued to think about prisms and pyramids to find an appropriate mathematical model for the lampshade. At last, he compared the lampshade with Christmas hats and drew a cone. Nuh Can's difficulty in spatial thinking and inappropriate visual representations of different surfaces and polyhedrons in his image prevented him from generating an appropriate mathematical model. However, Nuh Can stated that “its base will be a circle and it tapers [from the base] to the top” when he was defining a cone during the preliminary interview. It can be said that he could not adopt his concept definition to the problem situation in the process of obtaining a mathematical model. Nuh Can considered the cone and frustum of a cone when the researcher asked him to do so. However, he had several difficulties later in producing a correct solution for the problem.

On the other hand, Rukiye stated that “if we imagine the lampshade in our minds, it becomes a cone. It looks like it was cut in half or anywhere. So the lampshade is a frustum of a cone” while she was drawing a frustum of a cone to solve the same problem. Rukiye’s explanations and her drawing in Figure 7 were examples of the positive effects of mathematically appropriate concept images on the process of transforming the situation model into a mathematical model. Rukiye, who could model the lampshade object in the problem text in line with the numerical data given, examined this situation model from a mathematical point of view. Rukiye’s concept images of a cone and the frustum of a cone made it easier for her to perform a mathematically correct interpretation of the situation model. Thus, she became more familiar with the problem situation and had no difficulty in moving on to the stage of planning the solution.

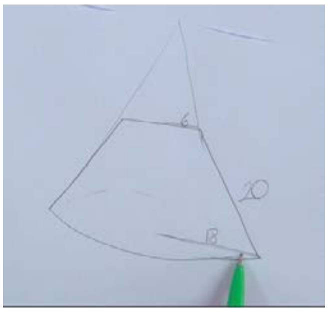

FIGURE 6. *Rukiye’s drawing of a frustum-shaped lampshade*

The reflections of the students’ concept images of geometric concepts to the mathematical modeling process were observed while they were working with the similarity concept in the “School Trip” problem. Realizing the parallelism between the ladders and that the triangles were similar triangles, Rukiye was able to generate an appropriate mathematical model by stating “I think there is a similarity here. When I extend this, I will obtain triangles” (see Figure 8). Joining the windows with a straight line, Rukiye created two triangles and established a relationship between the situation model and the concept of similarity. During the preliminary interviews, Rukiye expressed the concept of similarity as a relation between the sides of two shapes or more and preferred to explain the concept based on triangles. It can be said that her evoked concept image of the similarity positively affected the mathematizing process of the situation model.

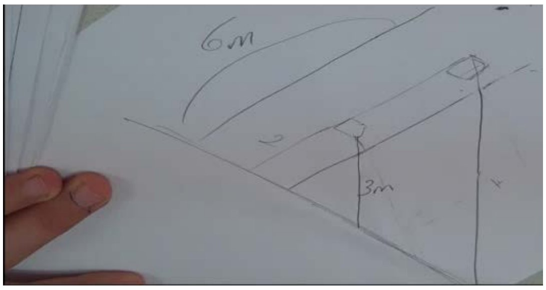

FIGURE 7. *Rukiye’s drawing of similar triangles obtained by the positions of the windows*

The students, who created a mathematical model by identifying which mathematical concepts corresponded to the real-life situation, performed mathematical analysis by applying field-specific knowledge and methods to these models. Findings of how their concept images influenced them in this process are presented in the next section.

Concept Images in Mathematical Analysis

The mathematical solution to a word problem is the product obtained as a result of a mathematical analysis of the problem situation (Verschaffel, De Corte, & Lasure, 1999). Students try to solve word problems by performing related operations with the help of mathematical models that they produced in the previous stage (Berry & Houston, 1995). Findings showed that students with mathematically appropriate concept images were more successful in identifying appropriate formulas, performing algebraic operations, and using mathematical notations correctly. When the situations in which students had difficulty in this process were scrutinized, it was discovered that they made errors in algebraic operations arising from their concept images of different mathematical concepts. They could not reason with the numerical results they found, which was observed to influence the students' tendency to produce incorrect solutions. For example, in the "Traffic Signs" problem, students were asked to find the slopes of two different ramps and determine which ramp would be more challenging for drivers. The ideas evoked in Seynur's concept image of rational numbers led her to have difficulty in finding the solution to the problem. The dialogue between the researcher and the student was as follows:

Researcher:	So you found the slopes of these two ramps. On which ramp will the driver have more difficulty?
Seynur:	On the first one.
Researcher:	Why on the first ramp?
Seynur:	Because it's steeper than the second one. The hill is steeper.
Researcher:	How did you know that?
Seynur:	10/26 is greater than 8/17 in numbers.

It can be said that the idea in the student's concept image of rational numbers, namely that the rational number with a bigger numerator and denominator is greater than the other rational number, led her to an incorrect solution. A concept image includes operations or procedures relating to a concept (Tall & Vinner, 1981). In another problem, Seynur's image of the square root concept influenced the results obtained while performing operations with square root numbers. Adding an integer with a square root number incorrectly, Seynur was affected adversely by the evoked concept image of the concept of square root at the stage of producing a solution to the problem (see Figure 9).

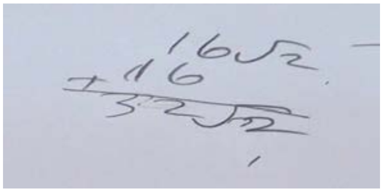

FIGURE 8. *Seynur's addition of a square root number with an integer*

Mathematical analyses of another slope problem provided some clues to how Seynur's concept image of the slope influenced her in finding a correct solution. Seynur, who could not develop an appropriate mathematical image for the concept of slope, was not able to solve the problem correctly even if she produced a mathematical model appropriate for the problem situation. The dialogue between her and the researcher is presented below:

Researcher:	How do you calculate the slope of this plank? [Refers to the line segment with the number 9 on it; see Figure 10]
Seynur:	Now here's a triangle. The lengths of the two sides are known, but the length of the other is unknown. To find the unknown side, I multiply the lengths of the two sides and divide the result in half [She does not apply what she says].
Researcher:	Is that how you find the length of a side of a triangle you don't know?

Seynur: Yes. Here, I would say the slope of the plank is 3/9, and reduce the fraction.
Researcher: Doesn't the other side matter?
Seynur: Yes, it does. If I knew the length of that side, then I would use 9 with that length.

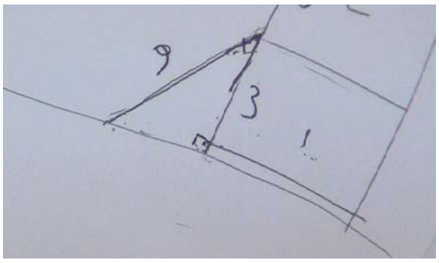

FIGURE 9. *Seynur's drawing in calculating the slope*

Seynur's focus was on the ratio of the opposite (or adjacent) side of a right triangle to the hypotenuse while she was talking about the slope of the plank. Later in the interview, Seynur calculated the third side's length for the triangle with the help of the Pythagorean Theorem. However, she indicated that the slope was 3/9 without changing her answer. When the researcher asked Seynur the reason for choosing these two sides, she replied: "Because these sides intersect." Although Seynur defined the slope as "the ratio of the vertical leg over the horizontal leg" during the preliminary interview, this definition was passive while she was analyzing her mathematical model. A mathematically inappropriate concept image prevented Seynur from making accurate mathematical operations and led her to produce an incorrect solution.

DISCUSSION and CONCLUSIONS

This study has investigated how eighth-grade students were affected by their concept images of mathematical concepts in solving geometric word problems. The students' success in producing an appropriate solution to each problem varied depending on the quality of their concept images. Students needed skills to use verbal, visual, and algebraic representations while solving geometric word problems and their concept images influenced their ability to use the representations effectively.

In understanding the problem, students who had inappropriate concept images of geometric concepts needed to read the problem many times to make sense of the data in the problem text. They had difficulty in understanding the *problem* due to redundantly focusing on the given *story* in the problem. The researcher had to explain the problem situations in her own words while she was working with these students, as she did in the "School Trip" problem during the interview conducted with Servet. These students could create a situation model for the given problem as the result of interventions that the researcher used during the interviews. This finding was similar to that of Ulusoy and Argun (2019), which showed that the interviewer's prompting questions during the interviews increased the students' awareness of and attention to the geometric word problems.

Blum and Borromeo Ferri (2009) state that students create a situation model after they imagine the problem situation mentally. The word problems in this study were based on geometric concepts. Since the figural and conceptual components of geometrical objects are not separable in geometrical reasoning (Fischbein, 1993), students needed to use visual representations of these concepts to imagine the problem situation. It was found that students who had inadequate images of the geometric concepts given in the problem situation had difficulty in creating situation models containing appropriate visual representations. For example, students' evoked concept images for the concepts of prism and pyramid affected them in creating an appropriate tent model for the related problem. This finding, which indicates that the richness of students' concept images of geometric shapes will positively affect the visualization of problem data, supports Yakimanskaya (1991) and Arcavi (2003). The inconsistencies between the students' concept definitions and concept images were another remarkable finding. Rösken and

Rolka (2007) state that “definitions play a marginal role in students’ learning” (p. 181), and students’ concept images, instead of their concept definitions, prevail in problem solving. Yanik (2014) supports this idea by indicating that middle school students depended more on their concept images in solving geometric translation problems. He found that students’ concept definitions were inconsistent with the formal concept definition of the transformation. In this study, the students’ concept definitions remained passive in understanding the problem unless they were consistent with their concept images. Students whose concept images were not consistent with their concept definitions needed help and more time to create a situation model for the problem.

In the “School Trip” problem, it was found that the students’ concept images of direction were limited to only horizontal and vertical ones. This made us think about the role of prototypes in developing rich and comprehensive concept images. Students often fall under the influence of prototypes when learning concepts, and they define concepts through prototypes (Pitta-Pantazi, Christou, & Zachariades, 2007). Since some of the features of prototypes do not fully coincide with the formal definitions of mathematical concepts (Tall & Bakar, 1992), students make mistakes when attempting to solve problems using prototypes. For example, Nuh Can’s expressions revealed that he thought objects would be in the same direction only when placed horizontally next to each other. Although he stated that the windows might be in a vertical direction when the researcher asked him to consider other options, he had to spend considerable time to create a situation model representing the windows in the same vertical direction. In this problem, other students associated the concept of direction with horizontal and vertical positioning, too. Pitta-Pantazi et al. (2007) state that “students often reject an instance as an exemplar of a concept because the instance lacks the attributes of the prototype” (p. 303). The students’ ignorance of other direction options might be because directions other than vertical and horizontal lack the attributes of their prototypes. In this study, prototypes, which affect students’ concept images (Levenson, Tirosh, & Tsamir, 2011; Okazaki & Fujita, 2007), influenced students in creating situation models including geometric concepts. This finding coincides with the idea proposed by Mariotti (1995) that images in geometry may prevail over concepts.

The students attempted to transform situation models into mathematical models in the stage of mathematical modeling. Verschaffel et al. (2002) draw attention to the fact that a solution could be reached through a mathematical model in solving word problems. Students with mathematically inadequate concept images had difficulty in identifying the mathematical concept to generate an appropriate mathematical model for the solution. On the other hand, students with rich concept images could easily use concepts as a tool in the stage of modeling. This was evident in the answers given by the students who wanted to generate a mathematical model in the “Art Class” problem. While Nuh Can’s concept image of different polyhedrons and surfaces hindered him in his efforts to obtain a correct mathematical model, Rukiye’s images of a cone and the frustum of a cone facilitated and accelerated the modeling process. Nuh Can’s concept definition, which was consistent with the formal definition of the concept, remained inactive in the stage of mathematical modeling. This also supports Mariotti’s (1995) idea that it is difficult for students to make a connection between images and concepts in geometry. It can be said that concept images were critical in determining which mathematical concepts corresponded to the real-life situation in geometric word problems.

After generating a mathematical model, students proceed to the stage of mathematical analysis, in which they try to find a solution to a problem through their mathematical models with the help of their mathematical knowledge (Berry & Houston, 1995). At this stage, the students tried to identify the necessary formulas and operations and to produce a solution. However, the students’ poor knowledge of the procedures in their concept images affected their mathematical results, especially when performing algebraic operations. For example, Seynur’s concept image of square root led her to an incorrect solution when performing operations with numbers. In another problem, Seynur’s evoked concept image of slope prevented her from reaching an appropriate solution to the problem.

Viholainen (2008) states that a concept image includes different "conceptions, representations and mental images concerning the meaning and properties of the concept and relationships to other concepts" (p. 235). The students used visual representations of geometric concepts rather than their definitions while trying to solve geometric word problems. Since a concept image is a personal cognitive structure, it can develop as a result of different experiences and become a new structure (Tall & Vinner, 1981). Considering that students' understandings of concepts are influenced by the contents of textbooks, classroom activities, and daily experiences, teachers need to present comprehensive examples while introducing mathematical concepts. As De Bock, Neyens, and Van Dooren (2017) state, enabling students to explore different representations of mathematical concepts and underlining the similarities and differences of representations by going beyond the prototype examples strengthen students' views of these concepts' mathematical properties. Because there is no direct access to mathematical objects, it is crucial to use different representations of concepts in teaching mathematics (Duval, 2006). Students must understand each of the representations of mathematical objects and relate these representations to each other during mathematical activities (Hoffman, 2006). In this study, Rukiye, who had concept images to which she could relate different representations of mathematical concepts, was able to solve geometric word problems more easily. Such findings support the study of De Bock, Van Dooren, and Verschaffel (2015), who highlight "the need for drawing sufficient instructional attention to representations, to discuss strengths and weaknesses of various representational forms, to match representations with each other, and to link them to realistic situations" in mathematics teaching (p. 64).

Additionally, the students had difficulty in stages of geometric word problem solving when their concept images were not coherent with their concept definitions. Although these students verbalized definitions that were consistent with formal definitions of the concepts, they used inappropriate parts of their concept images while solving the problems. They could not succeed in geometric word problems for which they needed to relate the definitions of these concepts to real-life situations. In this vein, geometric word problems were good tools to reveal students' concept images or to understand how they constructed a concept. Dickerson and Pitman (2016) indicate that students' written definitions are weak descriptors of their concept images and it will not be appropriate to rely on written definitions when it is desired to identify students' concept images. While interpreting students' concept images, mathematics teachers should consider that it is important not only to express concept definitions consistently with formal definitions but also to produce meaningful solutions to word problems that include these concepts.

REFERENCES

Abrams, J. P. (2001). Teaching mathematical modeling and the skills of representation. In A. A. Cuoco (Ed.). *The roles of representation in school mathematics (2001 Yearbook)* (pp. 269–282). Reston, VA: NCTM.

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics, 52,* 215-241.

Arslan, Ç. & Altun, M. (2007). Learning to solve non-routine mathematical problems. *Elementary Education Online*, *6*(1), 50-61.

Asman, D., & Markovits, Z. (2009). Elementary school teachers' knowledge and beliefs regarding non-routine problems. *Asia Pacific Journal of Education*, *29*(2), 229-249.

Bayazit, İ. (2013). İlköğretim 7. ve 8. sınıf öğrencilerinin gerçek yaşam problemlerini çözerken sergiledikleri yaklaşımlar ve kullandıkları strateji ve modellerin incelenmesi. *Educational Sciences: Theory & Practice, 13*(3), 1903–1927.

Berry, J., & Houston, K. (1995). *Mathematical modelling*. Oxford: Butterworth-Heinemann.

Bingölbali, E., & Monaghan, J. (2008). Concept image revisited. *Educational Studies in Mathematics*, *68*(1), 19-35.

Bingölbali, E. (2016). Kavram tanımı ve kavram imajı. In E. Bingölbali, S. Aslan, & İ. Ö. Zembat (Eds.), *Matematik eğitiminde teoriler* (pp. 135-148). Ankara: Pegem Akademi.

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modeling, applications, and links to other subjects: State, trends, and issues in mathematics education. *Educational Studies in Mathematics, 22*(1), 37–68.

Blum, W. & Borromeo Ferri, R. (2009). Mathematical modelling: Can it be taught and learnt? *Journal of Mathematical Modelling and Application, 1*(1), 45–58.

Chacko, I. (2004). Solution of real-world and standard problems by primary and secondary school students: A Zimbabwean example. *African Journal of Research in Mathematics, Science and Technology Education, 8 (2),* 91-103.
Chapman, O. (2006). Classroom practices for context of mathematics word problems. *Educational Studies in Mathematics, 62,* 211-230.
Çelik, D., & Güler, M. (2013). İlköğretim 6. sınıf öğrencilerinin gerçek yaşam problemlerini çözme becerilerinin incelenmesi. *Ziya Gökalp Eğitim Fakültesi Dergisi*, *20,* 180-195.
De Bock, D., Neyens, D., & Van Dooren, W. (2017). Student's ability to connect function properties to different types of elementary functions: An emprical study on the role of external representations. *International Journal of Science and Mathematics Education, 15*(5), 939-955.
De Bock, D., Van Dooren, W., & Verschaffel, L. (2015). Students' understanding of proportional, inverse proportional, and affine functions: Two studies on the role of external representations. *International Journal of Science and Mathematics Education, 13(1)*, 47–69.
De Corte, E., Greer, B., & Verschaffel, L. (1996). Mathematics learning and teaching. In D. Berliner & R. Calfee (Eds.), *Handbook of educational psychology* (pp. 491-549). New York: Macmillan.
Dickerson, D.S., & Pitman, D.J. (2016). An examination of college mathematics majors' understandings of their own written definitions. *Journal of Mathematical Behavior*, *41,* 1-9.
Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1-2), 103-131.
Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics, 24*(2), 139–162.
Galbraith, P., & Stillman, G. (2001). Assumptions and context. Pursuing their role in modelling activity. In J. F. Matos, W. Blum, S. K. Houston, & S. P. Carreira (Eds.), *Modelling and mathematics education* (pp. 300–310). Chichester: Horwood.
Gkoris, E., Depaepe, F., & Verschaffel, L. (2013). Investigating the gap between real world and school word problems. A comparative analysis of the authenticity of word problems in the old and the current mathematics textbooks for the 5th grade of elementary school in Greece. *The Mediterranean Journal for Research in Mathematics Education, 12*(1–2), 1–22.
Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517–545). Mahwah: Lawrence Erlbaum Associates, Inc.
Graeber, A. O., & Campbell, P. F. (1993). Misconceptions about multiplication and division. *Arithmetic Teacher, 40*(7), 408–411.
Greer, B. (1993). The modeling perspective on wor(l)d problems. *Journal of Mathematical Behavior, 12*, 239-250.
Hiebert, J., & LeFevre, P. (1986). Conceptual and procedural knowledge in mathematics: an introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1–27). Hillsdale: Lawrence Erlbaum Associates, Inc.
Hoffman, M. H. G. (2006). What is a "semiotic perspective", and what could it be? Some comments on the contributions to this special issue. *Educational Studies in Mathematics, 61*(1-2), 279-291.
Hoogland, K., Koning, J., Bakker, A., Pepin, B., & Gravemeijer, K. (2018). Changing representation in contextual mathematical problems from descriptive to depictive: The effect on students' performance. *Studies in Educational Evaluation, 58,* 122-131.
Karataş, İ., & Güven, B. (2010). Ortaöğretim öğrencilerinin günlük yaşam problemlerini çözebilme becerilerinin belirlenmesi. *Erzincan Eğitim Fakültesi Dergisi*, *12* (1), 201-217.
Kertil, M., Çetinkaya, B., Erbaş, A. K., & Çakıroğlu, E. (2016). Matematik eğitiminde matematiksel modelleme. In E. Bingölbali, S. Aslan, & İ. Ö. Zembat (Eds.), *Matematik eğitiminde teoriler* (pp. 540-563). Ankara: Pegem Akademi.
Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics. Washington*, DC: National Academy Press.
Krulik, S., & Rudnick, J. A. (1988). *Problem solving: A handbook for elementary school teachers*. Boston: Allyn and Bacon.
Levenson, E., Tirosh, D., & Tsamir, P. (2011). *Preschool Geometry.* Rotterdam: Sense Publishers.
Mariotti, M. A. (1995). Images and concepts in Geometrical reasoning. In R. Sutherland & J. Mason (Eds.), *Exploiting mental imagery with computer in mathematics education* (pp. 97–116). Berlin: Springer.
Mayer, R. E., & Wittrock, M. C. (2006). Problem solving. In P. A. Alexander & P. H. Winne (Eds.), *Handbook of educational psychology* (2nd ed., pp. 287–303). Mahwah, NJ: Erlbaum.
Merriam, S. B. (2009). *Qualitative research: A guide to design and implementation. Revised and expanded from qualitative research and case study applications in education* (2nd ed.). San Francisco, CA: Jossey- Bass.
Mesquita, A. L. (1998). On conceptual obstacles linked with external representation in geometry. *The Journal of Mathematical Behaviour, 17*(2), 183-195.
Muller, E., & Burkhardt, H. (2007). Applications and modelling for mathematics. In W. Blum, P. Galbraith, H. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education—The 14th ICMI Study* (pp. 267–274). New York: Springer.

Okazaki, M., & Fujita, T. (2007). Prototype phenomena and common cognitive paths in the understanding of the inclusion relations between quadrilaterals in Japan and Scotland. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 41–48). Seoul: Seoul National University.

Palm, T. (2008). Impact of authenticity on sense making in word problem solving. *Educational Studies in Mathematics, 67*(1), 37-58.

Patton, M. Q. (2002). *Qualitative research and evaluation methods.* Thousand Oaks, CA: Sage.

Pitta-Pantazi, D., Christou, C., & Zachariades, T. (2007). Secondary school students' levels of understanding in computing exponents. *Journal of Mathematical Behavior, 26*, 301–311.

Polya, G. (1957). *How to solve it: A new aspect of mathematical method* (2nd ed.). Garden City, NY: Doubleday.

Reed, S. K. (1999). *Word problems: Research and curriculum reform.* Mahwah, NJ: Lawrence Erlbaum Associates.

Rösken, B., & Rolka, K. (2007). Integrating intuition: the role of concept image and concept definition for students' learning of integral calculus. *The Montana Mathematics Enthusiast, 3*, 181–204.

Schoenfeld, A. H. (1985). *Mathematical problem solving.* London: Academic Press.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334–370). New York: MacMillan.

Stacey, K. (2005). The place of problem solving in contemporary mathematics curriculum documents. *Journal of Mathematical Behavior, 24,* 341–350.

Tall, D. O., & Bakar, M. (1992). Students' mental prototypes for functions and graphs. *International Journal of Mathematics Education in Science and Technology*, *23*(1), 39–50.

Tall, D., & Vinner, S. (1981). Concept images and concept definitions in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151-169.

Ulusoy, F., & Argun, Z. (2019). Secondary school students' representations for solving geometric word problems in different clinical interviews. *International Journal of Education in Mathematics Science and Technology*, *7*(1), 73-92.

Van De Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2010). *Elementary and middle school mathematics: Teaching developmentally (7th ed).* Boston: Allyn and Bacon/Pearson Education.

Verschaffel, L., Corte, E. D., & Lasure, S. (1994). Realistic considerations in mathematical modeling of school arithmetic word problems. *Learning and Instruction*, *4*(4), 273–294.

Verschaffel, L., De Corte, E., & Lasure, S. (1999). Children's conceptions about the role of real-world knowledge in mathematical modelling of school word problems. In W. Schnotz, S. Vosniadou, & M. Carretero (Eds.). *New perspectives on conceptual change* (pp. 175–189). Oxford: Elsevier.

Verschaffel, L., Greer, B., & De Corte, E. (2000). *Making sense of word problems*. Lisse: Swets and Zeitlinger.

Verschaffel, L., Greer, B., & De Corte, E. (2002). Everyday knowledge and mathematical modelling of school word problems. In K. Gravemeijer, R. Lehrer, B. van Oers, & L. Verschaffel (Eds.), *Symbolizing, modelling and tool use in mathematics education* (pp. 257–276). The Netherlands: Kluwer Academic Publishers

Verschaffel, L., Greer, B., Van Dooren, W., & Mukhopadhyay, S. (2009). *Words and worlds: Modelling verbal descriptions of situations.* Rotterdam: Sense Publishers.

Verschaffel, L., Schukajlow, S., Star, J., & Van Dooren, W. (2020). Word problems in mathematics education: a survey. *ZDM*, 1-16. DOİİİİ https://doi.org/10.1007/s11858-020-01130-4

Viholainen, A. (2008). Incoherence of a concept image and erroneous conclusion in the case of differentiability. *The Montana Mathematics Enthusiast, 5*(2-3), 231-248.

Vinner, S. (1983). Concept definition, concept image and the notion of function. *The International Journal of Mathematical Education in Science and Technology, 14*(3), 293–305.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*(4), 356–366.

Wong, W. K., Hsu, S. C., Wu, S. H., Lee, C. W., & Hsu, W. L. (2007). LIM-G: Learner-initiating instruction model based on cognitive knowledge for geometry word problem comprehension. *Computers & Education. 48*(4), 582–601.

Yakimanskaya, I. S. (1991). *Soviet Studies in Mathematics Education: Vol. 3. The development of spatial thinking in schoolchildren*. Reston, VA: National Council of Teachers of Mathematics.

Yanik, H. B. (2014). Middle-school students' concept images of geometric translations. *The Journal of Mathematical Behavior, 36*, 33–50.

Yin, R. K. (2014). *Case study research: Design and methods* (5th ed.). Thousand Oaks, CA: Sage.